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Research Article

A Symmetric Quadrature Formula of Degree Nine for Analytic Functions

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Abstract:

A new novel quadrature rule of precision 9 is developed for integration of analytic functions. For development of new quadrature rule Taylor's series plays a vital role. Richardson's extrapolation has been used in improving the degree of precision of Boole's precision from five to seven. Later the improved version of Boole's rule is combined with another rule of precision seven to produce quadrature rule of precision 9 using Taylor's expansion. In numerical computation four test examples has been adopted for investigating the efficiency of new quadrature rule and other classical quadrature rules. Approximated value, degree of precision and absolute error of newly developed quadrature rule along with other classical quadrature rules has been evaluated. It has been observed that the newly developed rule has less absolute error as compared to other rules which indicates that the new method is more efficient than the other methods. Also, absolute error of quadrature rules decreases with increase in degree of precision. Therefore, degree of precision varies inversely with absolute error of quadrature rules.

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1. Introduction

In many areas of science and engineering, it is often necessary to evaluate definite integrals of the form

$$I(f) = \int_a^b f(x)\, dx$$

where $f(x)$ may not possess an elementary antiderivative or may be given only by tabulated or experimental data. In such cases, numerical quadrature (also known as numerical integration) provides an efficient means of approximating the integral by replacing it with a finite weighted sum of function values at specific points within the interval $[a, b]$:

$$Q(f) = \sum_{i=0}^{n} w_i f(x_i)$$

where x_i are called the nodes (or abscissae) and w_i are the corresponding weights.

The central objective in constructing a quadrature rule is to choose appropriate nodes and weights so that the approximation $Q(f)$ is as close as possible to the exact integral $I(f)$. A quadrature rule is said to have degree of precision m (or order $m + 1$) if it integrates exactly all polynomials $f(x) = x^k$, for $k \le m$ but not necessarily for $k = m + 1$.Some classical quadrature rules include: Trapezoidal Rule, Simpson's $\frac{1}{3}rd$ rule, Boole's Rule, Gaussian Quadrature. The accuracy of any quadrature formula depends on both the smoothness of $f(x)$ and the order of the rule. The error term, typically expressed in terms of higher-order derivatives of $f(x)$, provides a quantitative measure of deviation between $I(f)$and $Q(f)$.

In modern numerical analysis, quadrature rules have been extended beyond polynomial exactness to handle oscillatory, singular, and improper integrals, and even multidimensional domains. Techniques such as adaptive quadrature, extrapolation (e.g., Richardson's method**)**, and hybrid rules further enhance accuracy and efficiency. Thus, numerical quadrature forms a cornerstone of computational mathematics, providing a unifying framework for approximating integrals in diverse applications such as physics, fluid mechanics, electromagnetism, and engineering design.

In this study we focused on evaluating the following type of integral [1]

$$I(f) = \int_{z_0-h}^{z_0+h} f(z)\, dz \qquad (1)$$

where L is the directed line segment from the point $z_0 - h$ to $z_0 + h$ in the complex plane and $f(z)$ is analytic in certain domain Ω containing the line segment L. Some classic quadrature rules can be used in evaluating (1.1). These are discussed as follows.

Trapezoidal rule is the most common quadrature rule which is used worldwide is defined by [2]

$$\int_a^b f(x)\, dx \cong \left(\frac{b-a}{2}\right)[f(a) + f(b)]$$

with error $E(T) = -\frac{(b-a)^3}{12} f''(\eta)\, , \eta \in [a, b]$.
Degree of precision of this rule is 1.

The transformed version of (1) using Trapezoidal rule is given by

$$I(f) = \int_{z_0-h}^{z_0+h} f(z)\, dz = h[f(z_0 - h) + f(z_0 + h)] \approx Q_T \qquad (2)$$

Another most commonly used quadrature rule is Simpson's $\frac{1}{3}rd$ rule which is given by[2]

$$\int_a^b f(x)\, dx \cong \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Where $h = \frac{b-a}{2}$ and error associated with this rule is $E\left(S_{\frac{1}{3}}\right) = -\frac{h^5}{90} f^{(4)}(\eta)\, , \eta \in [a, b]$.

Simpson's $\frac{1}{3}rd$ rule transforms (1) as follows

$$I(f) = \int_{z_0-h}^{z_0+h} f(z)\, dz = \frac{h}{3}[f(z_0 - h) + 4f(z_0) + f(z_0 + h)] \approx Q_{S_{\frac{1}{3}}}(f) \qquad (3)$$

Simpson's $\frac{3}{8}th$ rule is given by [2]

$$\int_{x_0}^{x_3} f(x)\, dx \cong \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

with error $E\left(S_{\frac{3}{8}}\right) = -\frac{3h^5}{80} f^{(4)}(\eta)\, , \eta \in [a, b]$.

Degree of precision of $Q_{S_{\frac{3}{8}}}(f)$ is 3 .

The transformed version of Eqn. (1) using Simpson's $\frac{3}{8}th$ rule is given by

$$I(f) = \int_{z_0-h}^{z_0+h} f(z)\,dz = \frac{h}{4}\left[f(z_0-h) + 3f\left(\frac{3z_0-h}{3}\right) + 3f\left(\frac{3z_0+h}{3}\right) + f(z_0+h)\right] \approx Q_{S_{\frac{3}{8}}}(f) \quad (4)$$

Gauss-Legendre 2- point quadrature rule of precision 3 is given by [3]

$$\int_{-1}^{1} f(x)\;dx \;\cong f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$

Gauss Legendre 2-point rule under transformation $z = z_0 + th$ (Lether) converts Eqn. (1) as follows

$$I(f) = \int_{z_0-h}^{z_0+h} f(z)\,dz = h\left[f\left(z_0 - \frac{h}{\sqrt{3}}\right) + f\left(z_0 + \frac{h}{\sqrt{3}}\right)\right] \approx Q_{GL2}(f) \qquad (5)$$

Another variant of Gauss quadrature rule is Gauss-Legendre 3-point rule of precision 5 defined by [3]

$$\int_{-1}^{1} f(x)\,dx \cong \frac{1}{9}\left[8f(0) + 5\left\{f\left(\sqrt{\frac{3}{5}}\right) + f\left(-\sqrt{\frac{3}{5}}\right)\right\}\right]$$

Transformed version of Eqn. (1) under Gauss-Legendre 3-point rule is follows

$$I(f) = \int_{z_0-h}^{z_0+h} f(z)\,dz = \frac{h}{9}\left[8f(z_0) + 5\left\{f\left(z_0 + \sqrt{\frac{3}{5}}h\right) + f\left(z_0 - \sqrt{\frac{3}{5}}h\right)\right\}\right] \approx Q_{GL3}(f) \quad (6)$$

A five-point quadrature rule of precision 5 named as Boole's quadrature rule $\left(Q_B(f)\right)$is represented by [2]

$$\int_a^b f(x)\;dx \;\cong \frac{2h}{45}\left[7f(a) + 32f(a+h) + 12f\left(\frac{a+b}{2}\right) + 32f(b-h) + 7f(b)\right] \qquad (7)$$

Where $h = \frac{b-a}{4}$ and error is $E_B = -\frac{8h^7}{945} f^6(\eta), \eta \in [a,b]$.

In 1996 Das et al [4] proposed a mixed quadrature rule of precision 5 for the evaluation of real definite integrals given by

$$\int_{-1}^{1} f(x)\,dx \;\cong \frac{1}{5}\left[2R_S(f) + 3R_{GL2}(f)\right]$$

$$R_S(f) = \frac{1}{3}\left[f(-1) + 4f(0) + f(1)\right]$$

$$R_{GL2}(f) = \left[f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)\right]$$

with error $E_{R_S}(f) = \frac{104}{315}.\frac{f^{(6)}(0)}{6!}$.

Transformed version of Eqn. (1) under above rule using transformation under transformation $z = z_0 + th$ (Lether) is as follows

$$I(f) = \int_{z_0-h}^{z_0+h} f(z)\,dz = \frac{1}{5}\left[2R_S(f) + 3R_{GL2}(f)\right] \approx Q_{R_S}(f) \qquad (8)$$

Where $R_S(f) = \frac{h}{3}\left[f(z_0-h) + 4f(z_0) + f(z_0+h)\right]$

$$R_{GL2}(f) = h\left[f\left(z_0 - \frac{h}{\sqrt{3}}\right) + f\left(z_0 + \frac{h}{\sqrt{3}}\right)\right]$$

In 2016 Jena et al developed a mixed quadrature rule $\left(Q_{WGL3}(f)\right)$ of precision 7 given by [5]

$$\int_{z_0-h}^{z_0+h} f(z)\ dz \cong R_{WGL3}(f) + E_{WGL3}(f) \quad (9)$$

Where $R_{WGL3}(f) = \frac{1}{511}[486R_W(f) + 25\ R_{GL3}(f)]$

Error associated with this rule is $E_{WGL3}(f) = \frac{8h^9}{657\times 8!} f^{(8)}(z_0)$.

In this study Richardson extrapolation [2] is introduced in enhancing the degree of precision of Boole's quadrature rule. After improvement of Boole's quadrature rule it is combined with mixed quadrature rule developed by Jena eta al [5] to produce a quadrature formula of precision 9. Taylor's series [6] plays a vital role in constructing new quadrature rules throughout this study.

Several recent studies have focused on the development and refinement of numerical quadrature rules for improved accuracy, stability, and applicability to complex integrands. Magalhaes et al. [7] introduced new quadrature formulas based on spline interpolation, which enhance smoothness and precision in integration, particularly for non-polynomial functions. Zhou et al. [8] extended quadrature formulations to geometric domains by constructing rules for Gregory quads and J. Kosinka & M. Barton [9] investigated on C^1 cubic Clough-Tocher macro-triangles, respectively, thereby contributing to surface and spline-based integration methods. Denich and Novati [10] developed quadrature techniques suitable for integrals involving oscillating functions, while Gautschi, Gori and Cascio [11] presented fundamental results on quadrature for rational functions. Mixed and hybrid quadrature schemes have also gained attention: Behera, Sethi, and Dash [12] proposed an open-type mixed rule combining Fejér and Gaussian quadratures, and Tripathy, Dash, and Baral [13] blended Lobatto and Gauss-Legendre three-point rules to achieve better endpoint and mid-interval accuracy. In a related development, T. Singh et al [14] formulated a novel quadrature rule tailored for analytic functions, extending the mixed-rule concept to higher precision. Further advancements were made by Rana [15], who proposed a harmonic and contra-harmonic mean-based rules, and by Reichel et al [16,17], who presented generalized and rational averaged Gauss quadrature rules with improved convergence behavior and broader applicability.

The structure of this paper is stated as follows: First section is introduction which deals with basic quadrature rules and other higher order quadrature rules that are necessary for this study. Preliminary ideas is the second section which is composed of basic definitions. Third section is mathematical formulation which deals with formation of new hybrid quadrature rule of precision 9. Numerical computation is the fourth section which deals with computation of computation degree of precision, absolute error and approximate value of different quadrature rules. Last section is conclusion.

2. Materials and Methods

2.1 Richardson's Extrapolation of Boole's Quadrature Rule

From Eqn. (1)

$$I(f) = \int_{z_0-h}^{z_0+h} f(z)\, dz$$

Using Boole's rule defined in Eqn. (7) in the above equation we have

$$I(f) = \int_{z_0-h}^{z_0+h} f(z)\, dz$$

$$\cong \frac{h}{45}\left[7f(z_0-h) + 32f\left(z_0-\frac{h}{2}\right) + 12f(z_0) + 32f\left(z_0+\frac{h}{2}\right) + 7f(z_0+h)\right] \approx Q(h) \ (10)$$

Applying Taylor's series to the functions present in the right hand side of the above integral we have,

$$Q(h) = 2hf(z_0) + \frac{h^3}{3}f''(z_0) + \frac{h^5}{60}f^{iv}(z_0) + \frac{h^7}{2160}f^{vi}(z_0) + \frac{19h^9}{241920}f^{viii}(z_0) + \frac{h^{11}}{11612160}f^{x}(z_0) + \cdots$$
(11)

Under the transformation $z = z_0 + th$ the integral defined in Equation (10) is simplified as follows

$$I(f) = h\int_{-1}^{1} f(z_0 + th)\ dt$$

Taylors Series expansion of $f(z_0 + th)$ about z_0 is as follows

$$f(z_0 + th) = \sum_{k=0}^{\infty} \frac{(th)^k}{k!} f^k(z_0)$$

Integrating above series term by term from $-1\ to\ 1$ we have,

$$I(f) = 2hf(z_0) + \frac{h^3}{3}f''(z_0) + \frac{h^5}{60}f^{iv}(z_0) + \frac{h^7}{2520}f^{vi}(z_0) + \frac{h^9}{181440}f^{viii}(z_0) + \frac{h^{11}}{19958400}f^{x}(z_0) + \cdots$$

Now $I(f) - Q(h) = -\frac{h^7}{15120}f^{iv}(z_0) - \frac{17h^9}{7257600}f^{viii}(z_0) - \frac{23h^{11}}{63866880}f^{x}(z_0) - \cdots$ (12)

Dividing the interval $[z_0 - h, z_0 + h\ ]$ into two equal halves $[z_0 - h, z_0]$ and $[z_0\ , z_0 + h\ ]$ we have from Eqn.(1)

$$I(f) = \int_{z_0-h}^{z_0+h} f(z)\, dz$$

$$= \int_{z_0-h}^{z_0} f(z)\, dz + \int_{z_0}^{z_0+h} f(z)\, dz$$

$$= \frac{h}{90}\left[7f(z_0 - h) + 32f\left(z_0 - \frac{3h}{4}\right) + 12f\left(z_0 - \frac{h}{2}\right) + 32f\left(z_0 - \frac{h}{4}\right) + 7f(z_0)\right] - \frac{8}{945}f^{vi}(\xi_1).\left(\frac{h}{2}\right)^7$$
$$+ \frac{h}{90}\left[7f(z_0) + 32f\left(z_0 + \frac{h}{4}\right) + 12f\left(z_0 + \frac{h}{2}\right) + 32f\left(z_0 + \frac{3h}{4}\right) + 7f(z_0 + h)\right] - \frac{8}{945}f^{vi}(\xi_2).\left(\frac{h}{2}\right)^7$$

$$= \frac{h}{90}\left[7\{f(z_0 - h) + f(z_0 + h)\} + 32\left\{f\left(z_0 - \frac{3h}{4}\right) + f\left(z_0 + \frac{3h}{4}\right)\right\} + 32\left\{f\left(z_0 - \frac{h}{4}\right) + f\left(z_0 + \frac{h}{4}\right)\right\}\right.$$
$$\left. + 12\left\{f\left(z_0 - \frac{h}{2}\right) + f\left(z_0 + \frac{h}{2}\right)\right\} + 14f(z_0)\right] - \frac{8}{945}.\left(\frac{h}{2}\right)^7.\left[f^{vi}(\xi_1) + f^{vi}(\xi_2)\right]$$

Where $\xi_1, \xi_2 \in\ [z_0 - h, z_0 + h\ ]$

Assuming $f^{vi}(\xi_1) + f^{vi}(\xi_2) \approx 2f^{vi}(\xi)$ and taking $Q\left(\frac{h}{2}\right) = \frac{h}{90}\left[7\{f(z_0 - h) + f(z_0 + h)\} + 32\left\{f\left(z_0 - \frac{3h}{4}\right) + f\left(z_0 + \frac{3h}{4}\right)\right\} + 32\left\{f\left(z_0 - \frac{h}{4}\right) + f\left(z_0 + \frac{h}{4}\right)\right\} + 12\left\{f\left(z_0 - \frac{h}{2}\right) + f\left(z_0 + \frac{h}{2}\right)\right\} + 14f(z_0)\right]$. Under these assumptions above integrals is written as follows

$$I(f) = \int_{z_0-h}^{z_0+h} f(z)\, dz = Q\left(\frac{h}{2}\right) - \frac{16}{945}\left(\frac{h}{2}\right)^7 f^{vi}(\xi) \qquad (12)$$

Now $I(f) - Q\left(\frac{h}{2}\right) = -\frac{h^7}{967680}f^{vi}(z_0) - \frac{257}{1857945600}f^{viii}(z_0) - \frac{2531}{653996851200}f^{x}(z_0) + \cdots$ (13)

Multiplying 15120 with Eqn. (11) and subtracting it from 967680 times of Eqn. (13) we have,

$$952560\, I(f) - 967680Q\left(\frac{h}{2}\right) + 15120\, Q(h) = -\frac{h^9}{9676800}f^{viii}(z_0) - \frac{103h^{11}}{30656102400}f^{x}(z_0) + \cdots$$

Which implies $I(f) = \frac{64Q\left(\frac{h}{2}\right) - Q(h)}{63} - \frac{h^9}{9676800}f^{viii}(z_0) - \frac{103h^{11}}{30656102400}f^{x}(z_0) + \cdots$

Therefore $\qquad I(f) = \frac{64Q\left(\frac{h}{2}\right) - Q(h)}{63} \approx R_Q(f) \qquad (14)$

This is Richardson's extrapolation of Boole's Quadrature rule.

2.2 Formation of Hybrid Quadrature Rule

From [5]

$$R_W(f) = 2hf(z_0) + \frac{h^3}{3}f''(z_0) + \frac{h^5}{60}f^{iv}(z_0) + \frac{7h^7}{17496}f^{vi}(z_0) + \frac{1307h^9}{220449600}f^{viii}(z_0) + \frac{713}{11904278400}f^{x}(z_0) + \cdots$$

$$R_{GL3}(f) = 2hf(z_0) + \frac{h^3}{3}f''(z_0) + \frac{h^5}{60}f^{iv}(z_0) + \frac{h^7}{3000}f^{vi}(z_0) + \frac{h^9}{280000}f^{viii}(z_0) + \frac{h^{11}}{42000000}f^{x}(z_0) + \cdots$$

$$R_{WGL3}(f) = 2hf(z_0) + \frac{h^3}{3}f''(z_0) + \frac{h^5}{60}f^{iv}(z_0) + \frac{h^7}{2520}f^{vi}(z_0) + \frac{11h^9}{1892160}f^{viii}(z_0) + \frac{5197h^{11}}{89404560000}f^{x}(z_0) + \cdots$$

Now $$I(f) - R_{WGL3}(f) = -\frac{h^9}{3311280}f^{viii}(z_0) - \frac{1973h^{11}}{245862540000}f^{x}(z_0) + \cdots \quad (15)$$

$$I(f) - R_Q(f) = -\frac{h^9}{9676800}f^{viii}(z_0) - \frac{103h^{11}}{30656102400}f^{x}(z_0) + \cdots \quad (16)$$

Multiplying 9676800 with Eqn. (16) and subtracting it from Eqn. (15) we have,

$$I(f) - \left[\frac{640R_Q(f) - 219R_{WGL3}(f)}{421}\right] = -\frac{21878212425653}{3679464657080200}h^{11}f^{x}(z_0) + \cdots \quad (17)$$

Therefore $$I(f) \cong \left[\frac{640R_Q(f) - 219R_{WGL3}(f)}{421}\right] \approx R_{QWGL3}(f) \quad (18)$$

The error term associated with $R_{QWGL3}(f)$ is $E_{QWGL3}(f) = \left[\frac{640E_Q(f) - 219E_{WGL3}(f)}{421}\right]$ (19)

Diagrammatic representation of $R_{QWGL3}(f)$ is drawn as follows:

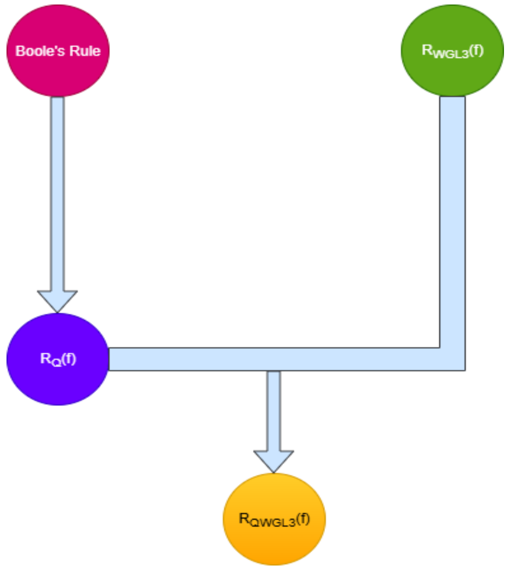

Figure 1: Hybrid Quadrature Rule

2.3 Error Analysis

Theorem. 1

If $f(z)$ is analytic in $[z_0 - h, z_0 + h]$. Then the error $E_{QWGL3}(f)$ associated with rule $R_{QWGL3}(f)$ is given by $\left|E_{QWGL3}(f)\right| = \frac{21878212425653}{3679464657080200} h^{11} f^{x}(z_0)$.

Proof:

From Eqn. (17) $E_{QWGL3}(f) = -\frac{21878212425653}{3679464657080200} h^{11} f^{x}(z_0)$

Which implies $\left|E_{QWGL3}(f)\right| = \frac{21878212425653}{3679464657080200} h^{11} f^{x}(z_0)$.

Theorem. 2

The error bound for truncation error of $R_{QWGL3}(f)$ is given by $\left|E_{QWGL3}(f)\right| \leq \frac{Mh^9}{63{,}65{,}520} |\eta_2 - \eta_1|$ where $M = \max_{z_0 - h \leq z \leq z_0 + h} \left|f^{ix}(z)\right|$.

Proof:

From Eqn. (19) $E_{QWGL3}(f) = \left[\frac{640E_Q(f) - 219E_{WGL3}(f)}{421}\right]$

From Eqn. (15) $E_{WGL3}(f) = -\frac{h^9}{3311280} f^{viii}(\eta_2), \eta_2 \in [z_0 - h, z_0 + h]$

From Eqn. (16) $E_Q(f) = -\frac{h^9}{9676800} f^{viii}(\eta_1), \eta_1 \in [z_0 - h, z_0 + h]$

$$E_{QWGL3}(f) = \frac{h^9}{421 \times 15120} \left[f^{viii}(\eta_2) - f^{viii}(\eta_1)\right]$$

Which implies $E_{QWGL3}(f) = \frac{h^9}{421 \times 15120} \int_{\eta_1}^{\eta_2} f^{ix}(z) dz$

$$\left|E_{QWGL3}(f)\right| \leq \frac{h^9}{421 \times 15120} \int_{\eta_1}^{\eta_2} \left|f^{ix}(z)\right| dz$$

$$= \frac{Mh^9}{421 \times 15120} \left| \int_{\eta_1}^{\eta_2} dz \right|$$

Therefore $\left|E_{QWGL3}(f)\right| \leq \frac{Mh^9}{421 \times 15120} |\eta_2 - \eta_1|$

It can be observed that the error will be less if η_2 *and* η_1 are closer to each other.

3. Results and Discussion

In this section, we conducted the numerical computations essential for our study. Utilizing Python software, we explored the approximate values of integrals, assessed the degree of precision, and examined the absolute errors associated with various quadrature rules. We ensured that all computed approximate values are accurate to 15 decimal places. To thoroughly evaluate the effectiveness of the new quadrature rule, we selected four integrals as case studies and compared its performance with that of classical quadrature rules. The examples provided below serve to illustrate the calculations of approximate values and their corresponding absolute errors.

1. $I_1 = \int_{-\frac{i}{4}}^{\frac{i}{4}} \frac{1}{z-1}\, dz$
2. $I_2 = \int_{-\frac{i}{2}}^{\frac{i}{2}} \cos z\; dz$
3. $I_3 = \int_{-i}^{i} e^{z}\, dz$
4. $I_4 = \int_{-\frac{i}{3}}^{\frac{i}{3}} \cosh z\; dz$

Table 1: Computation of degree of precision of quadrature rules

Quadrature Rules	Degree of Precision
Q_T	1
$Q_{S_{\frac{1}{3}}}(f)$	3
$Q_{S_{\frac{3}{3}}}(f)$	3
$Q_{GL_2}(f)$	3
$Q_{GL_3}(f)$	5
$Q(h)$	5
$Q_{R_S}(f)$	5
$Q_{WGL3}(f)$	7
$R_Q(f)$	7
$R_{QWGL3}(f)$	9

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Table 2: Computation of approximate values and absolute error of integrals

Quadrature Rules	Integrals	Approximate Value	Absolute Error
Q_T		-0.470588235294118i	0.019369090959610
$Q_{S_{\frac{1}{3}}}(f)$		-0.490196078431373i	0.000238752177645
$Q_{S_{\frac{3}{3}}}(f)$		-0.490060851926978i	0.000103525673250
$Q_{GL_2}(f)$		-0.489795918367347i	0.000161407886381
$Q_{GL_3}(f)$	I_1	-0.489959839357430i	$2.51310370197322 \times 10^{-6}$
$Q(h)$		-0.4899547511131222i	$2.575122506010000 \times 10^{-6}$
$Q_{R_S}(f)$		-0.489955982392957i	$1.34386077099702 \times 10^{-6}$
$Q_{WGL3}(f)$		- 0.489957366396379i	$4.01426509810676 \times 10^{-8}$
$R_Q(f)$		-0.4899573395543921i	$1.32901930016693 \times 10^{-8}$
$R_{QWGL3}(f)$		-0.489957325575540i	$6.7818800575381 \times 10^{-10}$
Q_T		1.1276259652063807j	0.085435354218886
$Q_{S_{\frac{1}{3}}}(f)$		1.0425419884021268j	0.000351377414632
$Q_{S_{\frac{3}{3}}}(f)$		1.0423472929601927j	0.000156681972698
$Q_{GL_2}(f)$		1.0419568234708350j	0.000233787516660
$Q_{GL_3}(f)$	I_2	1.0421901111537520j	$4.998337430617283 \times 10^{-7}$
$Q(h)$		1.0421911322798003j	$5.21292305322163 \times 10^{-7}$
$Q_{R_S}(f)$		1.0421908894433518j	$2.784558568169615 \times 10^{-7}$
$Q_{WGL3}(f)$		1.0421906115812647j	$5.937697000746311 \times 10^{-7}$
$R_Q(f)$		1.0421906111909767j	$2.034816759533 \times 10^{-10}$
$R_{QWGL3}(f)$		1.0421906109879536j	$4.576339307505 \times 10^{-13}$
Q_T		1.0806046117362795j	0.602337357879513
$Q_{S_{\frac{1}{3}}}(f)$		1.6935348705787600j	0.010592900962967
$Q_{S_{\frac{3}{3}}}(f)$		2.7755575616$\times 10^{-17}$+1.687586572406177j	0.004644602790384
$Q_{GL_2}(f)$		1.6758236553899861j	0.007118314225807
$Q_{GL_3}(f)$	I_3	1.6830035477269167j	$6.157811112372791 \times 10^{-5}$
$Q(h)$		1.6828781387363960j	$6.383087939698662 \times 10^{-5}$
$Q_{R_S}(f)$		1.6829081414654958j	$3.382815029717712 \times 10^{-5}$

$Q_{WGL3}(f)$		1.6829422636848246j	$2.940690315700323 \times 10^{-7}$
$R_Q(f)$		1.6829420696418276j	$1.0002603456626957 \times 10^{-7}$
$R_{QWGL3}(f)$		1.6829419687025962j	$9.131968514708433 \times 10^{-10}$
Q_T		0.6299712975431584j	0.024418096049145
$Q_{S_{\frac{1}{3}}}(f)$		0.6544348769588306j	$4.548336652665519 \times 10^{-5}$
$Q_{S_{\frac{3}{3}}}(f)$		0.6544095786552356j	$2.018506293166578 \times 10^{-5}$
$Q_{GL_2}(f)$		0.6543590445910790j	$3.0349001225005168 \times 10^{-5}$
$Q_{GL_3}(f)$	I_4	0.6543894225254678j	$3.0349001225005168 \times 10^{-5}$
$Q(h)$		0.6543893634698779j	$3.012242610900273 \times 10^{-8}$
$Q_{R_S}(f)$		0.6543893775381797j	$1.605412425220720 \times 10^{-8}$
$Q_{WGL3}(f)$		0.6543893936076023j	$1.5298318167822345 \times 10^{-11}$
$R_Q(f)$		0.6543893935975357j	$5.231703958941125 \times 10^{-12}$
$R_{QWGL3}(f)$		0.6543893935922992j	$4.773959005888173 \times 10^{-15}$

4. Conclusion

In this work, we introduce an innovative hybrid quadrature rule $R_{QWGL3}(f)$ with a precision level of nine for integration of analytic functions. This rule has been thoughtfully constructed by integrating a quadrature rule $Q_{WGL3}(f)$ [5] of precision seven with Richardson's extrapolation of Boole's rule. In the numerical computation section, we present approximate values of integrals, along with the corresponding absolute errors and degrees of precision for both the newly developed rule and various classical rules, facilitating a comprehensive comparison. Our findings suggest that the absolute error associated with the new quadrature rule is generally lower than that of several classical and higher-order methods. Furthermore, the degree of precision of this novel rule is higher than traditional methods. These computational observations indicate that the newly developed hybrid quadrature rule is more efficient when compared to traditional methods.

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References

[1] B.P. Acharya. Compound Birkhoff-Young rule for numerical integration of analytic functions. *Int. J. Math. Educ. Sci Technol*, 14(1): 101-103, 1983. https://doi.org/10.1080/0020739830140116

[2] R.L. Burden, J.D Faires. Numerical Analysis, Cengage Learning, Ninth Edition, 2011. https://faculty.ksu.edu.sa/sites/default/files/numerical_analysis_9th.pdf

[3] E. Suli and David F. Mayer. An Introduction to Numerical Analysis. Cambridge University press, 2003. http://www.statslab.cam.ac.uk/~lab85/resources/schedules2223.pdf

[4] R.N Das, G Pradhan. A mixed quadrature rule for approximate evaluation of real definite integrals, *International Journal of Mathematical education in Science*, 27: 279-283, 1996.
[5] S.R Jena, K. Meher & A.K. Paul. Approximation of analytic functions in adaptive environment. *Beni Suef Journal of Basic and Applied Sciences,* 5: 306-309, 2016. https://doi.org/10.1016/j.bjbas.2016.10.001
[6] Lin Ma, The Taylor's Theorem and Its Application, Highlights in Science, Engineering and Technology, 72: 1368-1372, 2023. https://doi.org/10.54097/98knf072
[7] P.A.A. Magalhaes, P.A.A. Magalhaes Junior, C. A Magalhaes, A.L.M.A Magalhaes. New Formulas of Numerical Quadrature Using Spline Interpolation, *Archieves of Computational Methods in Engineering,* 28; 553- 576, 2021. 10.1007/s11831-019-09391-3
[8] J. Zhoua, P. J. Barendrecht, M. Barton, J. Kosinka, Numerical quadrature for Gregory quads. *Applied Mathematics and Computation*, 453:1-14, 2023. https://www.cs.rug.nl/~jiri/papers/23ZhBaBaKo.pdf
[9] J. Kosinka, M. Barton. ˇ Gaussian quadrature for C1 cubic Clough-Tocher macro- triangles, *J. Comput. Appl. Math*, 351: 6–13, 2019. https://doi.org/10.1016/j.cam.2018.10.036
[10] E. Denich, P. Novati. Numerical quadrature for integrals involving oscillating functions. *Journal of Approximation Software*, 1:1-14, 2024.
[11] W. Gautschi, L. Gori, M.L. Lo Cascio. Quadrature rules for rational functions, Number. *Math,* 86: 617–633, 2000. https://doi.org/10.1007/PL00005412
[12] L. Reichel, Miodrag M. Spalevic. A new representation of generalized averaged Gauss quadrature rules, *Applied Numerical Mathematics,* 165:614–619, 2021. https://doi.org/10.1016/j.apnum.2020.11.016
[13] A. K. Tripathy, R. B. Dash and A. Baral. A mixed quadrature rule blending Lobatto and Gauss-Legendre 3-point rule for approximate evaluation of real definite integrals, *Int. J. Computing Science and Mathematics*, 6:366–377, 2015. https://doi.org/10.1504/IJCSM.2015.071809
[14] T. Singh, D.K. Behera. R.K. Saeed & S.A. Edalapanah. A novel quadrature rule for integration of analytic functions, *J. Mech. Cont. & Math. Sci.*, 20(2): 28-38, 2025. https://www.journalimcms.org/journal/a-novel-quadrature-rule-for-integration-of-analytic-functions/
[15] Rana, K. Harmonic Mean and Contra-Harmonic Mean Derivative-Based Closed Newton-Cotes Quadrature. *Integr. J. Res. Arts Humanit.* 2:55–61, 2022. https://doi.org/10.55544/ijrah.2.3.36
[16] L. Reichel, Miodrag M. Spalevic. A new representation of generalized averaged Gauss quadrature rules, *Applied Numerical Mathematics,* 165: 614–619, 2021. https://doi.org/10.1016/j.apnum.2020.11.016
[17] L. Reichela, Miodrag M. Spalevićb, Jelena D. Tomanovićb. Rational Averaged Gauss Quadrature Rules, 2: 379–389, 2020. https://doi.org/10.2298/FIL2002379R