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Research article

MetaHyperGraphs, MetaSuperHyperGraphs, and Iterated MetaGraphs: Modeling Graphs of Graphs, Hypergraphs of Hypergraphs, Superhypergraphs of Superhypergraphs, and Beyond

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Abstract

Graph theory studies mathematical structures composed of vertices and edges to model relationships and connectivity [1, 2]. Hypergraphs extend traditional graphs by allowing *hyperedges* that connect more than two vertices simultaneously [3]. Superhypergraphs further enrich this concept by introducing iterated powerset layers, enabling hierarchical and self-referential connections among hyperedges [4, 5]. A MetaGraph is a graph whose vertices are themselves graphs, with edges representing specified relations between those graphs. In this paper, we formally define the hypergraph analogue (MetaHyperGraph) and the superhypergraph analogue (MetaSuperHyperGraph) of MetaGraphs, and provide a concise discussion of their characteristics and illustrative applications. We also introduce iterative constructions such as the Iterated MetaGraph, representing a "graph of graphs of . . . of graphs," and briefly explore their properties and potential uses.

1. Preliminaries

This section collects the basic notions used throughout the paper.

1.1. SuperHyperGraph

A graph models relationships using vertices and edges, representing connections or interactions among entities in mathematics, computer science, and networks [1, 2]. A hypergraph extends an ordinary graph by allowing *hyperedges* that may join more than two vertices, which makes it suitable for modeling multiway relations in many areas [3, 6–11]. Known related concepts include Directed HyperGraphs[12, 13], Soft HyperGraphs[14, 15], Fuzzy HyperGraphs[16–18], and Neutrosophic HyperGraphs[19–23].

A *SuperHyperGraph* augments this idea by organizing vertices and edges through iterated powersets, thereby enabling hierarchical and self-referential linkages; see, e.g., [24–28]. Beyond theory, SuperHyperGraphs have seen use in applications such as molecular and network modeling and signal processing [29–34]. Throughout, the parameter n in the n-th (non-)empty powerset and in an n-SuperHyperGraph is a nonnegative integer.

Definition 1.1 (Base set)**.** *A* base set *S is the underlying universe from which all objects are formed. Formally,*

$$S = \{\, x \mid x \text{ belongs to the specified domain} \,\}.$$

All elements of constructions such as $\mathscr{P}(S)$ and $\mathscr{P}_n(S)$ are ultimately drawn from S.

Definition 1.2 (Powerset)**.** *[35] For a set S, the* powerset *$\mathscr{P}(S)$ is the family of all subsets of S:*

$$\mathscr{P}(S) = \{\, A \mid A \subseteq S \,\}.$$

Definition 1.3 (Hypergraph [3, 36])**.** *A* hypergraph *is a pair $H = (V(H), E(H))$ where*

- *$V(H)$ is a nonempty set of vertices;*
- *$E(H) \subseteq \mathscr{P}^*(V(H))$ is a set of nonempty subsets of $V(H)$ (the hyperedges).*

In this work we consider only finite hypergraphs.

Example 1.4 (Real-life Hypergraph: Project Teams in a Company)**.** *Let the vertices be employees*

$$V = \{\text{Alice, Bob, Chloe, Dan}\}.$$

Define three project teams as hyperedges

$$e_1 = \{\text{Alice}, \text{Bob}, \text{Dan}\}, \quad e_2 = \{\text{Bob}, \text{Chloe}\}, \quad e_3 = \{\text{Alice}, \text{Chloe}, \text{Dan}\}.$$

Then the hypergraph is $H = (V, E)$ with $E = \{e_1, e_2, e_3\} \subseteq \mathscr{P}^*(V)$.

Basic counts.

$$|V| = 4, \qquad |E| = 3, \qquad |e_1| = 3,\ |e_2| = 2,\ |e_3| = 3\ (\textit{non-uniform}).$$

Vertex degrees. *For $v \in V$, $\deg_H(v) = |\{e \in E : v \in e\}|$.*

$$\deg_H(\text{Alice}) = 2, \quad \deg_H(\text{Bob}) = 2, \quad \deg_H(\text{Chloe}) = 2, \quad \deg_H(\text{Dan}) = 2.$$

Incidence matrix. *With vertex order* (Alice, Bob, Chloe, Dan) *and edge order* (e_1, e_2, e_3)*,*

$$M(H) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

2-section (clique expansion). *Two employees are adjacent iff they appear together in some team. The induced simple graph has edges*

$$\{\text{A},\text{B}\}, \{\text{A},\text{C}\}, \{\text{A},\text{D}\}, \{\text{B},\text{C}\}, \{\text{B},\text{D}\}, \{\text{C},\text{D}\},$$

i.e., the complete graph K_4. Co-membership counts (as edge weights) are

$$w(\text{A},\text{B}) = 1,\ w(\text{A},\text{C}) = 1,\ w(\text{A},\text{D}) = 2,\ w(\text{B},\text{C}) = 1,\ w(\text{B},\text{D}) = 1,\ w(\text{C},\text{D}) = 1.$$

This model captures multiway team interactions that ordinary graphs (pairwise only) cannot.

Definition 1.5 (n-th powerset)**.** *(cf. [37–40]) For a set H and $n \geq 1$, define recursively*

$$\mathscr{P}_1(H) := \mathscr{P}(H), \qquad \mathscr{P}_{n+1}(H) := \mathscr{P}\big(\mathscr{P}_n(H)\big).$$

The n-th nonempty powerset *$\mathscr{P}_n^*(H)$ is defined analogously by*

$$\mathscr{P}_1^*(H) := \mathscr{P}^*(H), \qquad \mathscr{P}_{n+1}^*(H) := \mathscr{P}^*\big(\mathscr{P}_n^*(H)\big),$$

where $\mathscr{P}^*(H) := \mathscr{P}(H) \setminus \{\emptyset\}$.

Example 1.6 (A real-life reading of the n-th powerset: baskets → promotions → campaigns)**.** *Let the base set of* products *be*

$$H = \{\text{Milk, Bread, Eggs}\}.$$

Then the first powerset $\mathscr{P}_1(H) = \mathscr{P}(H)$ (all possible shopping baskets*) is*

$$\mathscr{P}_1(H) = \{\ \emptyset,\ \{\text{Milk}\},\ \{\text{Bread}\},\ \{\text{Eggs}\},\ \{\text{Milk}, \text{Bread}\},\ \{\text{Milk}, \text{Eggs}\},\ \{\text{Bread}, \text{Eggs}\},\ \{\text{Milk}, \text{Bread}, \text{Eggs}\}\ \}.$$

An element of the second powerset $\mathscr{P}_2(H) = \mathscr{P}\big(\mathscr{P}_1(H)\big)$ is a promotion*, i.e. a* set of baskets*. For instance,*

$$\mathscr{F}_1 := \{\ \{\text{Milk}\},\ \{\text{Bread}, \text{Eggs}\}\ \}, \qquad \mathscr{F}_2 := \{\ \emptyset,\ \{\text{Milk}, \text{Bread}, \text{Eggs}\}\ \}.$$

Since each listed basket belongs to $\mathscr{P}_1(H)$, we indeed have $\mathscr{F}_1, \mathscr{F}_2 \in \mathscr{P}_2(H)$.
An element of the third powerset $\mathscr{P}_3(H) = \mathscr{P}\big(\mathscr{P}_2(H)\big)$ is a campaign*, i.e. a* set of promotions*. For example,*

$$\mathscr{C} := \{\ \mathscr{F}_1,\ \mathscr{F}_2\ \}\ \in\ \mathscr{P}_3(H),$$

because $\mathscr{F}_1, \mathscr{F}_2 \in \mathscr{P}_2(H)$.

$\mathscr{P}_1(H)$: all baskets a customer could buy. $\mathscr{P}_2(H)$: a retailer's promotion, specified as a collection of eligible baskets. $\mathscr{P}_3(H)$: a marketing campaign, specified as a collection of promotions. This realizes the recursion $\mathscr{P}_{n+1}(H) = \mathscr{P}(\mathscr{P}_n(H))$ in a concrete setting.

Definition 1.7 (n-SuperHyperGraph)**.** *[4, 41, 42]*
Let V_0 be a finite base vertex set. Define the iterated powerset by

$$\mathscr{P}^0(V_0) := V_0, \qquad \mathscr{P}^{k+1}(V_0) := \mathscr{P}\big(\mathscr{P}^k(V_0)\big) \quad (k \geq 0),$$

*and write $\mathscr{P}^*(X) := \mathscr{P}(X) \setminus \{\emptyset\}$ for the family of all nonempty subsets of a set X. An (undirected, simple) n-*SuperHyperGraph *on V_0 is a pair*

$$\mathrm{SHG}^{(n)} = (V, E)$$

such that

$$V \subseteq \mathscr{P}^n(V_0) \ \text{ is finite}, \qquad E \subseteq \mathscr{P}^*(V).$$

*Elements of V are called n-*supervertices, *and each $e \in E$ is a (nonempty) n-*superedge, *i.e. a finite subset of V. Equivalently, $\mathrm{SHG}^{(n)}$ is a (simple) hypergraph whose vertex set is a finite subset of $\mathscr{P}^n(V_0)$; the incidence relation $\iota \subseteq V \times E$ is given by $v\,\iota\,e \iff v \in e$.*

Example 1.8 (A 2-SuperHyperGraph)**.** *Let the base set be $V_0 = \{a, b, c\}$. Then $\mathscr{P}(V_0)$ consists of*

$$\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{a,b\},\ \{a,c\},\ \{b,c\},\ \{a,b,c\},$$

and $\mathscr{P}^2(V_0) = \mathscr{P}\big(\mathscr{P}(V_0)\big)$.
Define three 2-supervertices (each is a subset of $\mathscr{P}(V_0)$, hence an element of $\mathscr{P}^2(V_0)$):

$$S_1 := \big\{\{a\},\{b,c\}\big\}, \qquad S_2 := \big\{\emptyset,\{b\}\big\}, \qquad S_3 := \big\{\{a,c\}\big\}.$$

Set the vertex set and hyperedge set as

$$V := \{S_1, S_2, S_3\} \subseteq \mathscr{P}^2(V_0), \qquad E := \big\{\, e_1, e_2 \,\big\} \subseteq \mathscr{P}^*(V),$$

with

$$e_1 := \{S_1, S_2\}, \qquad e_2 := \{S_2, S_3\}.$$

Then $\mathrm{SHG}^{(2)} := (V, E)$ is a (simple, undirected) 2-SuperHyperGraph.
Verification (universe membership):

$$\{a\},\{b,c\},\emptyset,\{b\},\{a,c\} \in \mathscr{P}(V_0) \implies S_1, S_2, S_3 \in \mathscr{P}^2(V_0),$$

and e_1, e_2 are nonempty subsets of V, so $E \subseteq \mathscr{P}^*(V)$.

Example 1.9 (A 3-SuperHyperGraph)**.** *Let $V_0 = \{a, b, c\}$ as above. First choose elements of $\mathscr{P}^2(V_0)$ (i.e., subsets of $\mathscr{P}(V_0)$):*

$$A_1 := \big\{\{a\},\{b\}\big\}, \qquad A_2 := \big\{\{a,b\},\{c\}\big\}, \qquad A_3 := \big\{\emptyset\big\}.$$

Since each $A_i \subseteq \mathscr{P}(V_0)$, we have $A_1, A_2, A_3 \in \mathscr{P}^2(V_0)$.
Define three 3-supervertices (each is a subset of $\mathscr{P}^2(V_0)$, hence an element of $\mathscr{P}^3(V_0)$):

$$U_1 := \{A_1, A_3\}, \qquad U_2 := \{A_2\}, \qquad U_3 := \{A_1, A_2\}.$$

Set

$$V' := \{U_1, U_2, U_3\} \subseteq \mathscr{P}^3(V_0), \qquad E' := \big\{\, f_1, f_2 \,\big\} \subseteq \mathscr{P}^*(V'),$$

with

$$f_1 := \{U_1, U_2\}, \qquad f_2 := \{U_2, U_3\}.$$

Then $\mathrm{SHG}^{(3)} := (V', E')$ is a valid (simple, undirected) 3-SuperHyperGraph.
Verification (universe membership):

$$A_1, A_2, A_3 \in \mathscr{P}^2(V_0) \implies U_1, U_2, U_3 \in \mathscr{P}^3(V_0),$$

and f_1, f_2 are nonempty subsets of V', hence $E' \subseteq \mathscr{P}^*(V')$.

1.2. MetaGraph(Graph of Graph)

A MetaGraph is a graph whose vertices are themselves graphs, with edges representing specified relations between those graphs (cf.[43, 43–47]).

Definition 1.10 (Metagraph (graph of graphs)). *(cf.[48]) Fix a nonempty universe $\mathfrak{G}$ of finite graphs (undirected, loopless by default) and a nonempty family of binary relations*

$$\mathscr{R} \subseteq \mathscr{P}\big(\mathfrak{G}\times\mathfrak{G}\big).$$

A metagraph over $(\mathfrak{G},\mathscr{R})$ *is a directed, labelled multigraph*

$$M = (V,E,s,t,\lambda)$$

with

$$V\subseteq\mathfrak{G}, \qquad s,t : E\to V, \qquad \lambda : E\to\mathscr{R},$$

satisfying the incidence constraint

$$\forall e\in E:\ \big(s(e),t(e)\big)\in\lambda(e).$$

Elements of V are meta-vertices *(each is a graph $G\in\mathfrak{G}$). For $e\in E$ with $\lambda(e)=R$, we write $s(e)\xrightarrow{R}t(e)$ and call e a* meta-edge*. If $\mathscr{R}=\{R\}$ is a singleton, labels may be omitted. If every $R\in\mathscr{R}$ is symmetric, M can be viewed as an undirected labelled multigraph.*

Example 1.11 (Real-life MetaGraph: cross-citing departments (graph of graphs)). *Let $\mathfrak{G}$ be the class of finite directed acyclic* citation graphs *(vertices=papers, edges=citations). Consider three department graphs*

$$\begin{aligned} G_{\text{CS}}:&\ V=\{c_1,c_2,c_3\},\ E=\{c_2\to c_1,\ c_3\to c_2\},\\ G_{\text{Bio}}:&\ V=\{b_1,b_2\},\ E=\{b_2\to b_1\},\\ G_{\text{Math}}:&\ V=\{m_1,m_2\},\ E=\{m_2\to m_1\}.\end{aligned}$$

Let X be the set of observed cross-department *citations:*

$$X=\{c_3\to b_1,\ c_1\to m_1,\ b_2\to c_2,\ m_2\to c_1\}.$$

Define the directed relation R_τ on $\mathfrak{G}$ by the numeric threshold

$$c(G,H) := \big|\{(p,q)\in V(G)\times V(H):\ p\to q\in X\}\big|, \qquad (G,H)\in R_\tau \iff c(G,H)\geq\tau.$$

With $\tau=1$, the counts are

$$c(G_{\text{CS}},G_{\text{Bio}})=1, \quad c(G_{\text{CS}},G_{\text{Math}})=1, \quad c(G_{\text{Bio}},G_{\text{CS}})=1, \quad c(G_{\text{Math}},G_{\text{CS}})=1,$$

all others $=0$. Hence the metagraph over $(\mathfrak{G},\{R_1\})$ is

$$M=(V,E,s,t,\lambda), \quad V=\{G_{\text{CS}},G_{\text{Bio}},G_{\text{Math}}\},$$

with (edge set and maps made explicit)

$$\begin{aligned} &E=\{e_1,e_2,e_3,e_4\}, \quad \lambda(e_i)=R_1\ \ (i=1,\dots,4),\\ s(e_1)=G_{\text{CS}},\ t(e_1)=G_{\text{Bio}}, &\qquad s(e_2)=G_{\text{CS}},\ t(e_2)=G_{\text{Math}},\\ s(e_3)=G_{\text{Bio}},\ t(e_3)=G_{\text{CS}}, &\qquad s(e_4)=G_{\text{Math}},\ t(e_4)=G_{\text{CS}}.\end{aligned}$$

By construction, each incidence satisfies $(s(e_i),t(e_i))\in R_1$ because the corresponding $c(\cdot,\cdot)=1$.

2. Review and Result of this Paper

This section provides an explanation and discussion of the review and results presented in this paper.

2.1. Iterated MetaGraph(Graph of Graph of ... of Graph)

An Iterated MetaGraph is a graph whose vertices are metagraphs, recursively extending graph-of-graphs structure to multiple hierarchical levels.

Definition 2.1 (Unit metagraph embedding). *For $X\in\mathfrak{G}$ define the* unit metagraph

$$\mathsf{U}(X) := \big(\{X\},\emptyset,\_,\_,\_\big).$$

This gives an injective map $\mathsf{U}:\mathfrak{G}\hookrightarrow\mathrm{Obj}\big(\mathrm{Meta}(\mathfrak{G},\mathscr{R})\big)$.

Example 2.2 (Unit metagraph embedding: a single project as a meta-vertex)**.** ***Real-life setting.*** *Let $\mathfrak{G}$ be the universe of finite directed* email graphs *inside a company, where a vertex is an employee and a directed edge $u \to v$ means "u sent an email to v".*
Consider one specific project's internal email graph

$$G_{\text{Phoenix}} = (V, E), \qquad V = \{p_1, p_2\}, \quad E = \{\, p_2 \to p_1 \,\}.$$

Unit embedding. *Its unit metagraph is*

$$\mathsf{U}(G_{\text{Phoenix}}) = \big(\, \{\, G_{\text{Phoenix}} \,\},\, \emptyset,\, \_,\, \_,\, \_\, \big),$$

i.e., a metagraph with the single meta-vertex G_{Phoenix} and no meta-edges.
Concreteness check (injectivity on this instance). *If another project $G_{\text{Orion}} \in \mathfrak{G}$ satisfies $G_{\text{Orion}} \neq G_{\text{Phoenix}}$, then*

$$\{G_{\text{Orion}}\} \neq \{G_{\text{Phoenix}}\} \quad \Longrightarrow \quad \mathsf{U}(G_{\text{Orion}}) \neq \mathsf{U}(G_{\text{Phoenix}}),$$

so U distinguishes distinct base graphs in this concrete scenario.

Definition 2.3 (Relation lifting)**.** *Given $\mathscr{R}$ on $\mathfrak{G}$, define its* lift *$\mathscr{R}^{\uparrow}$ on finite metagraphs over $(\mathfrak{G}, \mathscr{R})$ by*

$$\forall R \in \mathscr{R}, \quad (M_1, M_2) \in R^{\uparrow} \iff \exists x \in V(M_1),\, y \in V(M_2) :\ (x, y) \in R.$$

Set $\mathscr{R}^{\uparrow} := \{R^{\uparrow} : R \in \mathscr{R}\}$.

Example 2.4 (Relation lifting: departments related by one cross-project email)**.** ***Base universe and relation.*** *Let $\mathfrak{G}$ again be company email graphs. Define a cross-project email log*

$$X = \{\, s_1 \to r_1 \,\},$$

and for $G, H \in \mathfrak{G}$ put the count

$$c(G, H) := \big|\{(u, v) \in V(G) \times V(H) :\ u \to v \in X\}\big|.$$

Let the base relation be

$$(G, H) \in R_{\text{mail}} \iff c(G, H) \geq 1.$$

Concrete projects (graphs).

$$\begin{aligned} G_{\text{Sales}} &: \ V = \{s_1, s_2\},\ E = \{\, s_2 \to s_1 \,\}, \\ G_{\text{Support}} &: \ V = \{u_1, u_2\},\ E = \{\, u_2 \to u_1 \,\}, \\ G_{\text{Research}} &: \ V = \{r_1, r_2\},\ E = \{\, r_2 \to r_1 \,\}. \end{aligned}$$

From X we compute

$$c(G_{\text{Sales}}, G_{\text{Research}}) = 1, \qquad \textit{all other } c(\cdot, \cdot) = 0.$$

Hence

$$\big(G_{\text{Sales}}, G_{\text{Research}}\big) \in R_{\text{mail}}.$$

Metagraphs and the lifted relation. *Form two (finite) metagraphs over $(\mathfrak{G}, \{R_{\text{mail}}\})$:*

$$M_1 :\ V(M_1) = \{G_{\text{Sales}}, G_{\text{Support}}\}, \quad E(M_1) = \emptyset,$$

$$M_2 :\ V(M_2) = \{G_{\text{Research}}\}, \quad E(M_2) = \emptyset.$$

The lift $R^{\uparrow}_{\text{mail}}$ satisfies

$$(M_1, M_2) \in R^{\uparrow}_{\text{mail}} \iff \exists x \in V(M_1),\, y \in V(M_2) :\ (x, y) \in R_{\text{mail}}.$$

Witness. *Take $x = G_{\text{Sales}}$, $y = G_{\text{Research}}$. Since $c(G_{\text{Sales}}, G_{\text{Research}}) = 1$, we have $(x, y) \in R_{\text{mail}}$, hence $(M_1, M_2) \in R^{\uparrow}_{\text{mail}}$.*

Definition 2.5 (Iterated object and relation universes)**.** *Define recursively for $t \in \mathbb{N}_0$:*

$$\mathfrak{G}^{(0)} := \mathfrak{G}, \qquad \mathscr{R}^{(0)} := \mathscr{R},$$

$$\mathfrak{G}^{(t+1)} := \Big\{ \textit{finite metagraphs over } \big(\mathfrak{G}^{(t)}, \mathscr{R}^{(t)}\big) \Big\}, \qquad \mathscr{R}^{(t+1)} := \big(\mathscr{R}^{(t)}\big)^{\uparrow}.$$

Example 2.6 (Iterated universes: projects $\to$ departments $\to$ companies)**.** ***Step*** $t = 0$ ***(base objects and relations).*** *Let* $\mathfrak{G}^{(0)} = \mathfrak{G}$ *be project email graphs and* $\mathscr{R}^{(0)} = \{R_{\text{mail}}\}$*, where* R_{mail} *is defined from the cross-project log*

$$X = \{a_2 \to b_1,\ b_1 \to c_1,\ a_1 \to c_2\}.$$

Concrete project graphs:

$$G_A:\ V = \{a_1, a_2\},\ E = \{a_2 \to a_1\},$$
$$G_B:\ V = \{b_1, b_2\},\ E = \{b_2 \to b_1\},$$
$$G_C:\ V = \{c_1, c_2\},\ E = \{c_2 \to c_1\}.$$

Cross counts derived from X*:*

$$c(G_A, G_B) = 1, \quad c(G_B, G_C) = 1, \quad c(G_A, G_C) = 1, \quad \textit{others} = 0.$$

Thus

$$(G_A, G_B),\ (G_B, G_C),\ (G_A, G_C) \in R_{\text{mail}}.$$

Step $t = 1$ ***(metagraphs over*** $(\mathfrak{G}^{(0)}, \mathscr{R}^{(0)})$***).*** *Define two department metagraphs (*$\mathfrak{G}^{(1)}$ *contains such objects):*

$$M_{\text{Dept1}} = (V, E, s, t, \lambda), \quad V = \{G_A, G_B\},\ E = \{e_{AB}\},\ s(e_{AB}) = G_A,\ t(e_{AB}) = G_B,\ \lambda(e_{AB}) = R_{\text{mail}},$$

which is valid because $(G_A, G_B) \in R_{\text{mail}}$*; and*

$$M_{\text{Dept2}} = (V', E', s', t', \lambda'), \quad V' = \{G_C\},\ E' = \emptyset.$$

The lifted relation family is $\mathscr{R}^{(1)} = \{R^{\uparrow}_{\text{mail}}\}$ *with*

$$(M, N) \in R^{\uparrow}_{\text{mail}} \iff \exists x \in V(M), y \in V(N):\ (x, y) \in R_{\text{mail}}.$$

Verification. *Using* $x = G_B \in V(M_{\text{Dept1}})$ *and* $y = G_C \in V(M_{\text{Dept2}})$ *and* $c(G_B, G_C) = 1$*, we have*

$$\big(M_{\text{Dept1}}, M_{\text{Dept2}}\big) \in R^{\uparrow}_{\text{mail}}.$$

Step $t = 2$ ***(metagraphs over*** $(\mathfrak{G}^{(1)}, \mathscr{R}^{(1)})$***).*** *Now* $\mathfrak{G}^{(2)}$ *consists of finite metagraphs whose vertices are department metagraphs. Define two companies:*

$$\mathsf{CoX}:\ V = \{M_{\text{Dept1}}\},\ E = \emptyset; \qquad \mathsf{CoY}:\ V = \{M_{\text{Dept2}}\},\ E = \emptyset.$$

The relation family lifts again to $\mathscr{R}^{(2)} = \{(R^{\uparrow}_{\text{mail}})^{\uparrow}\}$*. By definition,*

$$(\mathsf{CoX}, \mathsf{CoY}) \in (R^{\uparrow}_{\text{mail}})^{\uparrow} \iff \exists M \in V(\mathsf{CoX}), N \in V(\mathsf{CoY}):\ (M, N) \in R^{\uparrow}_{\text{mail}}.$$

Witness. *Take* $M = M_{\text{Dept1}}$*,* $N = M_{\text{Dept2}}$*; we verified* $(M_{\text{Dept1}}, M_{\text{Dept2}}) \in R^{\uparrow}_{\text{mail}}$ *above, hence*

$$(\mathsf{CoX}, \mathsf{CoY}) \in (R^{\uparrow}_{\text{mail}})^{\uparrow}.$$

Conclusion. *This realizes the recursion*

$$\mathfrak{G}^{(t+1)} = \{\textit{finite metagraphs over } (\mathfrak{G}^{(t)}, \mathscr{R}^{(t)})\}, \qquad \mathscr{R}^{(t+1)} = (\mathscr{R}^{(t)})^{\uparrow},$$

with explicit numeric witnesses at each level ($t = 0, 1, 2$*).*

Definition 2.7 (Iterated MetaGraph of depth t)**.** *For* $t \in \mathbb{N}_0$*, an* iterated metagraph of depth t *is a metagraph*

$$M^{(t)} = (V^{(t)}, E^{(t)}, s^{(t)}, t^{(t)}, \lambda^{(t)})$$

over $\big(\mathfrak{G}^{(t)}, \mathscr{R}^{(t)}\big)$*, i.e.,* $V^{(t)} \subseteq \mathfrak{G}^{(t)}$*,* $\lambda^{(t)}: E^{(t)} \to \mathscr{R}^{(t)}$ *and*

$$\forall e \in E^{(t)}:\ \big(s^{(t)}(e), t^{(t)}(e)\big) \in \lambda^{(t)}(e).$$

Remark 2.8. *Depth* 0 *vertices are graphs; depth* 1 *vertices are metagraphs of graphs; depth* 2 *vertices are metagraphs whose vertices are metagraphs of graphs; etc. Edges are always certified by the corresponding lifted relation at that depth.*

Example 2.9 (Real-life Iterated MetaGraph: universities built from departmental metagraphs)**.** *Fix the same thresholded relation R_τ on departmental citation graphs as above.*
University A *has two departments with internal paper-level graphs*

$$G^{\mathsf{A}}_{\mathrm{CS}}: \{a_c^2 \to a_c^1\}, \qquad G^{\mathsf{A}}_{\mathrm{Bio}}: \{a_b^2 \to a_b^1\},$$

and within-A *cross-citations*

$$X_{\mathsf{A}} = \{\, a_c^2 \to a_b^1 \,\} \;\Rightarrow\; c\big(G^{\mathsf{A}}_{\mathrm{CS}}, G^{\mathsf{A}}_{\mathrm{Bio}}\big) = 1.$$

Thus the (depth-0) metagraph for A *is*

$$M_{\mathsf{A}} = (\{G^{\mathsf{A}}_{\mathrm{CS}}, G^{\mathsf{A}}_{\mathrm{Bio}}\}, \{e_{\mathsf{A}}\}, s, t, \lambda), \quad s(e_{\mathsf{A}}) = G^{\mathsf{A}}_{\mathrm{CS}},\; t(e_{\mathsf{A}}) = G^{\mathsf{A}}_{\mathrm{Bio}},\; \lambda(e_{\mathsf{A}}) = R_1.$$

University B *has departments*

$$G^{\mathsf{B}}_{\mathrm{CS}}: \textit{ no internal edge}, \qquad G^{\mathsf{B}}_{\mathrm{Math}}: \{b_m^2 \to b_m^1\},$$

and within-B *cross-citations*

$$X_{\mathsf{B}} = \{\, b_c^1 \to b_m^1 \,\} \;\Rightarrow\; c\big(G^{\mathsf{B}}_{\mathrm{CS}}, G^{\mathsf{B}}_{\mathrm{Math}}\big) = 1,$$

so

$$M_{\mathsf{B}} = (\{G^{\mathsf{B}}_{\mathrm{CS}}, G^{\mathsf{B}}_{\mathrm{Math}}\}, \{e_{\mathsf{B}}\}, s, t, \lambda), \quad s(e_{\mathsf{B}}) = G^{\mathsf{B}}_{\mathrm{CS}},\; t(e_{\mathsf{B}}) = G^{\mathsf{B}}_{\mathrm{Math}},\; \lambda(e_{\mathsf{B}}) = R_1.$$

Now define cross-university *citations*

$$X_{\mathsf{A}\to\mathsf{B}} = \{\, a_b^2 \to b_m^1 \,\}, \qquad X_{\mathsf{B}\to\mathsf{A}} = \{\, b_c^1 \to a_c^1 \,\}.$$

The lifted relation $R_1^{\uparrow}$ on metagraphs (as in the iterated construction) satisfies

$$\big(M_{\mathsf{A}}, M_{\mathsf{B}}\big) \in R_1^{\uparrow} \iff \exists G \in V(M_{\mathsf{A}}), H \in V(M_{\mathsf{B}}): \; (G,H) \in R_1.$$

Numerically,

$$c\big(G^{\mathsf{A}}_{\mathrm{Bio}}, G^{\mathsf{B}}_{\mathrm{Math}}\big) = 1 \Rightarrow \big(M_{\mathsf{A}}, M_{\mathsf{B}}\big) \in R_1^{\uparrow},$$

$$c\big(G^{\mathsf{B}}_{\mathrm{CS}}, G^{\mathsf{A}}_{\mathrm{CS}}\big) = 1 \Rightarrow \big(M_{\mathsf{B}}, M_{\mathsf{A}}\big) \in R_1^{\uparrow}.$$

Hence the iterated metagraph *(depth* 1*) of universities is*

$$\mathbf{M} = (V^{(1)}, E^{(1)}, s^{(1)}, t^{(1)}, \lambda^{(1)}), \quad V^{(1)} = \{M_{\mathsf{A}}, M_{\mathsf{B}}\},$$

$$E^{(1)} = \{E_{\mathsf{A}\to\mathsf{B}}, E_{\mathsf{B}\to\mathsf{A}}\}, \quad s^{(1)}(E_{\mathsf{A}\to\mathsf{B}}) = M_{\mathsf{A}},\; t^{(1)}(E_{\mathsf{A}\to\mathsf{B}}) = M_{\mathsf{B}},$$

$$s^{(1)}(E_{\mathsf{B}\to\mathsf{A}}) = M_{\mathsf{B}},\; t^{(1)}(E_{\mathsf{B}\to\mathsf{A}}) = M_{\mathsf{A}}, \quad \lambda^{(1)}(\cdot) = R_1^{\uparrow}.$$

Each edge satisfies the incidence constraint by the explicit counts $c(\cdot,\cdot) = 1$ shown above.

Example 2.10 (Iterated MetaGraph of depth 3: airlines → alliances → consortia → clusters)**.** *We build the structure level by level and verify every incidence numerically.*

Level 0 (base graphs: airline route graphs). *Let the set of airports be $\mathscr{A} = \{X,Y,Z,W,V\}$. Define five finite (undirected, simple) graphs, each representing an airline's direct-flight network:*

$$\begin{aligned} &G_{\mathrm{A1}}: V = \{X,Y\},\; E = \big\{\{X,Y\}\big\}, \quad G_{\mathrm{A2}}: V = \{Y,Z\},\; E = \big\{\{Y,Z\}\big\},\\ &G_{\mathrm{B1}}: V = \{W,X\},\; E = \big\{\{W,X\}\big\}, \quad G_{\mathrm{B2}}: V = \{Z,W\},\; E = \big\{\{Z,W\}\big\},\\ &G_{\mathrm{C1}}: V = \{V,W\},\; E = \big\{\{V,W\}\big\}. \end{aligned}$$

Let $\mathrm{name}(G) \in \{\mathrm{A1}, \mathrm{A2}, \mathrm{B1}, \mathrm{B2}, \mathrm{C1}\}$ *be the airline code.*

Codeshare relation at level 0. *Let the symmetric "codeshare list" be*

$$\mathscr{X} = \big\{\; (\mathrm{A1},\mathrm{A2}),\; (\mathrm{B1},\mathrm{B2}),\; (\mathrm{A1},\mathrm{B1}),\; (\mathrm{A2},\mathrm{B2}),\; (\mathrm{B1},\mathrm{C1}) \;\big\}.$$

Define

$$c(G,H) := \begin{cases} 1, & \big(\mathrm{name}(G), \mathrm{name}(H)\big) \in \mathscr{X} \textit{ or its swap},\\ 0, & \textit{otherwise}, \end{cases} \qquad (G,H) \in R_{\mathrm{cs}} \iff c(G,H) = 1.$$

Thus $R_{\mathrm{cs}} \subseteq \{G_{\mathrm{A1}}, G_{\mathrm{A2}}, G_{\mathrm{B1}}, G_{\mathrm{B2}}, G_{\mathrm{C1}}\}^2$ is a well-defined symmetric binary relation.

Depth 1 (metagraphs of airlines: alliances). *Form three metagraphs whose vertices are the airline graphs, with meta-edges labeled by* R_{cs}*:*

$$\begin{aligned} M_{\mathrm{A}} &= (V_{\mathrm{A}}, E_{\mathrm{A}}, s, t, \lambda), \quad V_{\mathrm{A}} = \{G_{\mathrm{A1}}, G_{\mathrm{A2}}\}, \\ &\qquad E_{\mathrm{A}} = \{e_{\mathrm{A}}\},\ s(e_{\mathrm{A}}) = G_{\mathrm{A1}},\ t(e_{\mathrm{A}}) = G_{\mathrm{A2}},\ \lambda(e_{\mathrm{A}}) = R_{\mathrm{cs}}; \\ M_{\mathrm{B}} &= (V_{\mathrm{B}}, E_{\mathrm{B}}, s, t, \lambda), \quad V_{\mathrm{B}} = \{G_{\mathrm{B1}}, G_{\mathrm{B2}}\}, \\ &\qquad E_{\mathrm{B}} = \{e_{\mathrm{B}}\},\ s(e_{\mathrm{B}}) = G_{\mathrm{B1}},\ t(e_{\mathrm{B}}) = G_{\mathrm{B2}},\ \lambda(e_{\mathrm{B}}) = R_{\mathrm{cs}}; \\ M_{\Gamma} &= (V_{\Gamma}, E_{\Gamma}, s, t, \lambda), \quad V_{\Gamma} = \{G_{\mathrm{C1}}\},\ E_{\Gamma} = \emptyset. \end{aligned}$$

Incidence check at depth 1. *We have*

$$c(G_{\mathrm{A1}}, G_{\mathrm{A2}}) = 1, \quad c(G_{\mathrm{B1}}, G_{\mathrm{B2}}) = 1,$$

hence $\big(s(e), t(e)\big) \in R_{\mathrm{cs}}$ *for* $e \in \{e_{\mathrm{A}}, e_{\mathrm{B}}\}$*, as required.*

Depth 2 (metagraph of alliances: consortia). *Lift* R_{cs} *to alliances by*

$$(M,N) \in R_{\mathrm{cs}}^{\uparrow} \iff \exists x \in V(M),\, y \in V(N) :\ (x,y) \in R_{\mathrm{cs}}.$$

Define the "consortia" metagraph

$$\mathbf{M}^{(2)} = (\mathbf{V}^{(2)}, \mathbf{E}^{(2)}, s^{(2)}, t^{(2)}, \lambda^{(2)}), \quad \mathbf{V}^{(2)} = \{M_{\mathrm{A}}, M_{\mathrm{B}}, M_{\Gamma}\},$$

with edges

$$\mathbf{E}^{(2)} = \{E_{\mathrm{A}\to\mathrm{B}},\ E_{\mathrm{B}\to\Gamma}\}, \quad \lambda^{(2)}(\cdot) = R_{\mathrm{cs}}^{\uparrow},$$

and

$$s^{(2)}(E_{\mathrm{A}\to\mathrm{B}}) = M_{\mathrm{A}},\ t^{(2)}(E_{\mathrm{A}\to\mathrm{B}}) = M_{\mathrm{B}}, \quad s^{(2)}(E_{\mathrm{B}\to\Gamma}) = M_{\mathrm{B}},\ t^{(2)}(E_{\mathrm{B}\to\Gamma}) = M_{\Gamma}.$$

Incidence check at depth 2. *Witness for* $E_{\mathrm{A}\to\mathrm{B}}$*: take* $x = G_{\mathrm{A1}} \in V(M_{\mathrm{A}})$*,* $y = G_{\mathrm{B1}} \in V(M_{\mathrm{B}})$*, then* $c(x,y) = 1 \Rightarrow (x,y) \in R_{\mathrm{cs}}$*, hence* $(M_{\mathrm{A}}, M_{\mathrm{B}}) \in R_{\mathrm{cs}}^{\uparrow}$*. Witness for* $E_{\mathrm{B}\to\Gamma}$*:* $x = G_{\mathrm{B1}} \in V(M_{\mathrm{B}})$*,* $y = G_{\mathrm{C1}} \in V(M_{\Gamma})$*, so* $c(x,y) = 1$ *and* $(M_{\mathrm{B}}, M_{\Gamma}) \in R_{\mathrm{cs}}^{\uparrow}$*. By design there is* no *edge* $M_{\mathrm{A}} \to M_{\Gamma}$ *since* $c(G_{\mathrm{A1}}, G_{\mathrm{C1}}) = c(G_{\mathrm{A2}}, G_{\mathrm{C1}}) = 0$*.*

Depth 3 (metagraph of consortia: clusters). *Form two "cluster" vertices whose elements are consortia (i.e., vertices of* $\mathbf{M}^{(2)}$*):*

$$\mathscr{C}_{\mathrm{North}} := \big\{M_{\mathrm{A}}, M_{\mathrm{B}}\big\}, \qquad \mathscr{C}_{\mathrm{South}} := \big\{M_{\Gamma}\big\}.$$

Lift once more:

$$(\mathscr{C}, \mathscr{D}) \in R_{\mathrm{cs}}^{\uparrow\uparrow} \iff \exists M \in \mathscr{C},\, N \in \mathscr{D} :\ (M,N) \in R_{\mathrm{cs}}^{\uparrow}.$$

*Define the depth-*3 *metagraph*

$$\mathbf{M}^{(3)} = (\mathbf{V}^{(3)}, \mathbf{E}^{(3)}, s^{(3)}, t^{(3)}, \lambda^{(3)}), \quad \mathbf{V}^{(3)} = \big\{\mathscr{C}_{\mathrm{North}}, \mathscr{C}_{\mathrm{South}}\big\},$$

with a single edge $E_{\mathrm{North}\to\mathrm{South}}$ *labeled* $\lambda^{(3)} = R_{\mathrm{cs}}^{\uparrow\uparrow}$ *and*

$$s^{(3)}(E_{\mathrm{North}\to\mathrm{South}}) = \mathscr{C}_{\mathrm{North}}, \qquad t^{(3)}(E_{\mathrm{North}\to\mathrm{South}}) = \mathscr{C}_{\mathrm{South}}.$$

Incidence check at depth 3 (explicit witnesses). *Choose* $M = M_{\mathrm{B}} \in \mathscr{C}_{\mathrm{North}}$ *and* $N = M_{\Gamma} \in \mathscr{C}_{\mathrm{South}}$*. From the depth-*2 *verification,* $(M_{\mathrm{B}}, M_{\Gamma}) \in R_{\mathrm{cs}}^{\uparrow}$ *via the pair* $(G_{\mathrm{B1}}, G_{\mathrm{C1}})$ *with* $c = 1$*. Therefore* $(\mathscr{C}_{\mathrm{North}}, \mathscr{C}_{\mathrm{South}}) \in R_{\mathrm{cs}}^{\uparrow\uparrow}$*, and the incidence constraint for* $E_{\mathrm{North}\to\mathrm{South}}$ *holds.*

In summary, we have explicitly constructed an Iterated MetaGraph of depth 3*: airline route graphs (level 0)* $\to$ *alliances (depth 1)* $\to$ *consortia (depth 2)* $\to$ *clusters (depth 3), with all edges justified by concrete codeshare witnesses at each lift.*

Theorem 2.11 (Iterated MetaGraphs generalize MetaGraphs)**.** *Every metagraph* M *over* $(\mathfrak{G}, \mathscr{R})$ *is (canonically) isomorphic to an induced sub-metagraph of some depth-*1 *iterated metagraph over* $\big(\mathfrak{G}^{(1)}, \mathscr{R}^{(1)}\big)$*. In particular, the depth-*0 *case is exactly Definition of metagraph.*

Proof. Let $M = (V, E, s, t, \lambda)$ be a metagraph over $(\mathfrak{G}, \mathscr{R})$. Define a vertex map

$$\Phi_V : V \longrightarrow \mathfrak{G}^{(1)}, \qquad \Phi_V(X) := \mathsf{U}(X).$$

For each $e \in E$ with $\lambda(e) = R \in \mathscr{R}$ and $(s(e), t(e)) \in R$, create a depth-1 edge

$$\Phi_E(e) :\ \mathsf{U}\big(s(e)\big) \xrightarrow{R^{\uparrow}} \mathsf{U}\big(t(e)\big).$$

This is well-defined because

$$\big(\mathsf{U}(s(e)),\, \mathsf{U}(t(e))\big) \in R^{\uparrow} \iff \exists x \in \{s(e)\},\, y \in \{t(e)\} :\ (x,y) \in R \iff \big(s(e), t(e)\big) \in R,$$

using the definition of U and of $R^{\uparrow}$. Let $M^{(1)}$ be the depth-1 metagraph with vertex set $\Phi_V(V)$ and edge set $\Phi_E(E)$. Then (Φ_V, Φ_E) is a label-preserving metagraph isomorphism from M onto the induced sub-metagraph of $M^{(1)}$ on $\Phi_V(V)$, since

$$\lambda(e) = R \quad \Longrightarrow \quad \lambda^{(1)}\big(\Phi_E(e)\big) = R^{\uparrow},$$

and incidence is preserved by the equivalence above. Hence any (depth-0) metagraph embeds canonically into a depth-1 iterated metagraph. The assertion that depth 0 recovers the Definition is immediate from Definitions 2.5–2.7. □

2.2. MetaHyperGraph(HyperGraph of HyperGraph)

A MetaHyperGraph is a hypergraph whose vertices are themselves hypergraphs, with hyperedges encoding relations among these component hypergraphs.

Notation 2.12. *For a set X, write $\mathscr{P}_{\text{fin}}(X)$ for the family of all finite (possibly empty) subsets of X and $\mathscr{P}^*_{\text{fin}}(X) := \mathscr{P}_{\text{fin}}(X) \setminus \{\emptyset\}$.*

Definition 2.13 (Directed hypergraph)**.** *A* directed hypergraph *is a tuple $H = (V, E, T, Hd)$ with V a vertex set, E an edge set, and $T, Hd : E \to \mathscr{P}_{\text{fin}}(V)$ the tail/head maps such that $T(e) \cup Hd(e) \neq \emptyset$ for all $e \in E$.*

Definition 2.14 (MetaHyperGraph over $(\mathfrak{U}, \mathscr{R})$)**.** *Let $\mathfrak{U}$ be a nonempty universe of objects and let*

$$\mathscr{R} \subseteq \mathscr{P}\Big(\mathscr{P}_{\text{fin}}(\mathfrak{U}) \times \mathscr{P}_{\text{fin}}(\mathfrak{U})\Big)$$

be a nonempty family of admissible set–relations. *A* MetaHyperGraph *over $(\mathfrak{U}, \mathscr{R})$ is a labelled directed hypergraph*

$$\mathsf{M} = (V, E, T, Hd, \lambda)$$

with $V \subseteq \mathfrak{U}$, $T, Hd : E \to \mathscr{P}_{\text{fin}}(V)$, $\lambda : E \to \mathscr{R}$, satisfying the incidence constraint

$$\forall e \in E : \quad \big(T(e), Hd(e)\big) \in \lambda(e).$$

Vertices in V are meta-vertices. *If $\mathfrak{U}$ is the class of finite hypergraphs, we say that M is a* HyperGraph of HyperGraphs.

Remark 2.15. *Undirected MetaHyperGraphs are obtained by requiring each $R \in \mathscr{R}$ to be symmetric and identifying (T, Hd) with (Hd, T). Ordinary directed metagraphs (graphs of graphs) arise when every hyperedge is a pair of singletons; see Theorem 2.17.*

Example 2.16 (Real-life MetaHyperGraph: hospital departments sharing patients)**.** *Let each vertex be a finite (undirected) hypergraph whose vertices are* patient IDs *and whose hyperedges are* procedure sessions. *Consider three departmental hypergraphs:*

$$\begin{aligned} H_{\text{Rad}} &: V = \{p_1, p_2, p_3\}, \quad E = \{\{p_1, p_2\}, \{p_2, p_3\}\}, \\ H_{\text{Card}} &: V = \{p_2, p_4\}, \quad E = \{\{p_2\}, \{p_2, p_4\}\}, \\ H_{\text{Onc}} &: V = \{p_1, p_2, p_5\}, \quad E = \{\{p_1, p_2\}, \{p_2, p_5\}\}. \end{aligned}$$

Define the admissible set–relation R_{share} on finite families of departmental hypergraphs by

$$(S, T) \in R_{\text{share}} \iff \exists x \text{ (patient ID) such that } \forall H \in S \cup T,\ \exists e \in E(H) :\ x \in e.$$

Form a MetaHyperGraph $\mathsf{M} = (V, E, T, Hd, \lambda)$ over $(\{H_{\text{Rad}}, H_{\text{Card}}, H_{\text{Onc}}\}, \{R_{\text{share}}\})$ with

$$V = \{H_{\text{Rad}}, H_{\text{Card}}, H_{\text{Onc}}\}, \qquad E = \{e_1\}, \qquad T(e_1) = \{H_{\text{Rad}}, H_{\text{Card}}\}, \quad Hd(e_1) = \{H_{\text{Onc}}\},$$

and $\lambda(e_1) = R_{\text{share}}$. Verification of the incidence constraint:

$$(S, T) = \big(T(e_1), Hd(e_1)\big) = \big(\{H_{\text{Rad}}, H_{\text{Card}}\}, \{H_{\text{Onc}}\}\big).$$

Take $x = p_2$. Then

$$p_2 \in \{p_1, p_2\} \in E(H_{\text{Rad}}), \quad p_2 \in \{p_2\} \in E(H_{\text{Card}}), \quad p_2 \in \{p_1, p_2\} \in E(H_{\text{Onc}}),$$

so $(S, T) \in R_{\text{share}}$, hence $(T(e_1), Hd(e_1)) \in \lambda(e_1)$ as required.

Theorem 2.17 (MetaHyperGraphs generalize MetaGraphs)**.** *Fix a universe $\mathfrak{U}$ and a family of binary relations $\mathscr{R}_2 \subseteq \mathscr{P}(\mathfrak{U} \times \mathfrak{U})$. Embed $\mathscr{R}_2$ into set–relations by*

$$\iota : \mathscr{R}_2 \longrightarrow \mathscr{P}\Big(\mathscr{P}_{\text{fin}}(\mathfrak{U}) \times \mathscr{P}_{\text{fin}}(\mathfrak{U})\Big), \qquad \iota(R) := \big\{ (\{x\}, \{y\}) \mid (x, y) \in R \big\}.$$

Then the full subcategory of MetaHyperGraphs over $(\mathfrak{U}, \iota(\mathscr{R}_2))$ with edges satisfying $|T(e)| = |Hd(e)| = 1$ is (canonically) equivalent to the category of labelled directed metagraphs *over $(\mathfrak{U}, \mathscr{R}_2)$.*

Proof. Given a MetaHyperGraph $\mathsf{M} = (V, E, T, Hd, \lambda)$ with $|T(e)| = |Hd(e)| = 1$, define a metagraph

$$M = (V, E, s, t, \lambda_2)$$

by $s(e)$ the unique element of $T(e)$, $t(e)$ the unique element of $Hd(e)$, and $\lambda_2(e)$ the unique $R \in \mathscr{R}_2$ with $\lambda(e) = \iota(R)$. The incidence constraint gives

$$\big(T(e), Hd(e)\big) \in \lambda(e) = \iota(R) \iff \big(\{s(e)\}, \{t(e)\}\big) \in \iota(R) \iff \big(s(e), t(e)\big) \in R,$$

hence M is a metagraph over $(\mathfrak{U}, \mathscr{R}_2)$.
Conversely, given a metagraph $M = (V, E, s, t, \lambda_2)$ over $(\mathfrak{U}, \mathscr{R}_2)$, set

$$\mathsf{M} = (V, E, T, Hd, \lambda), \qquad T(e) := \{s(e)\},\ Hd(e) := \{t(e)\},\ \lambda(e) := \iota\big(\lambda_2(e)\big).$$

Then $(T(e), Hd(e)) \in \iota(\lambda_2(e))$ holds iff $(s(e), t(e)) \in \lambda_2(e)$, so M is a MetaHyperGraph. These two constructions are mutually inverse on objects and morphisms, giving the claimed equivalence. □

Theorem 2.18 (MetaHyperGraphs generalize HyperGraphs)**.** *Let X be a nonempty set and define the trivial admissible relation*

$$\mathbf{U} := \mathscr{P}^*_{\text{fin}}(X) \times \{\emptyset\} \subseteq \mathscr{P}_{\text{fin}}(X) \times \mathscr{P}_{\text{fin}}(X),$$

and $\mathscr{R} := \{\mathbf{U}\}$. The assignment

$$\Phi: \textit{MetaHyperGraphs over } (X, \mathscr{R}) \textit{ with } Hd(e) = \emptyset \longleftrightarrow \textit{(undirected) hypergraphs on } X$$

given by

$$\Phi: (V{=}X, E, T, Hd \equiv \emptyset, \lambda \equiv \mathbf{U}) \longmapsto H = (X, E_H), \; E_H := \{T(e) : e \in E\},$$

is a bijection on isomorphism classes. In particular, every undirected hypergraph arises as a specialization of a MetaHyperGraph.

Proof. For any specialized MetaHyperGraph as stated, the incidence constraint reads $(T(e), \emptyset) \in \mathbf{U}$, which is tautologically true for all nonempty $T(e) \in \mathscr{P}^*_{\text{fin}}(X)$. Thus $E_H := \{T(e) : e \in E\} \subseteq \mathscr{P}^*_{\text{fin}}(X)$ defines a hypergraph $H = (X, E_H)$.
Conversely, given any hypergraph $H = (X, E_H)$, define $E := E_H$, $T(e) := e$, $Hd(e) := \emptyset$ and $\lambda(e) := \mathbf{U}$; then $(T(e), \emptyset) \in \mathbf{U}$ holds by definition, so $\mathsf{M} = (X, E, T, Hd, \lambda)$ is a MetaHyperGraph over $(X, \mathscr{R})$. Isomorphisms clearly match under these maps, yielding a bijection on isomorphism classes. ☐

2.3. Iterated MetaHyperGraph(HyperGraph of HyperGraph of ... of HyperGraph)

An Iterated MetaHyperGraph is a hypergraph whose vertices are meta-hypergraphs, recursively building hypergraph-of-hypergraph structures over multiple hierarchical depths.

Definition 2.19 (Lift of set–relations for iteration)**.** *Let $\mathfrak{U}$, $\mathscr{R}$ be as above. For $R \in \mathscr{R}$ and MetaHyperGraphs $\mathsf{M}_1 = (V_1, \ldots)$, $\mathsf{M}_2 = (V_2, \ldots)$ over $(\mathfrak{U}, \mathscr{R})$, define the* lift $R^{\Uparrow}$ *on $\mathscr{P}_{\text{fin}}(\{\mathsf{M}\}) \times \mathscr{P}_{\text{fin}}(\{\mathsf{M}\})$ by*

$$(S, T) \in R^{\Uparrow} \iff \exists M \in S, \; N \in T, \; \exists A \in \mathscr{P}_{\text{fin}}(V(M)), \; B \in \mathscr{P}_{\text{fin}}(V(N)) : \; (A, B) \in R.$$

Set $\mathscr{R}^{\Uparrow} := \{R^{\Uparrow} : R \in \mathscr{R}\}$.

Definition 2.20 (Iterated universes)**.** *Define recursively for $t \in \mathbb{N}_0$:*

$$\mathfrak{U}^{(0)} := \mathfrak{U}, \quad \mathscr{R}^{(0)} := \mathscr{R},$$

$$\mathfrak{U}^{(t+1)} := \big\{\textit{finite MetaHyperGraphs over } (\mathfrak{U}^{(t)}, \mathscr{R}^{(t)})\big\},$$

$$\mathscr{R}^{(t+1)} := \big(\mathscr{R}^{(t)}\big)^{\Uparrow}.$$

Definition 2.21 (Iterated MetaHyperGraph of depth t)**.** *For $t \in \mathbb{N}_0$, an* Iterated MetaHyperGraph (IMHG) of depth t *is a MetaHyperGraph*

$$\mathsf{M}^{(t)} = (V^{(t)}, E^{(t)}, T^{(t)}, Hd^{(t)}, \lambda^{(t)})$$

over $\big(\mathfrak{U}^{(t)}, \mathscr{R}^{(t)}\big)$, i.e., $V^{(t)} \subseteq \mathfrak{U}^{(t)}$, $\lambda^{(t)} : E^{(t)} \to \mathscr{R}^{(t)}$, and

$$\forall e \in E^{(t)} : \; \big(T^{(t)}(e), Hd^{(t)}(e)\big) \in \lambda^{(t)}(e).$$

Remark 2.22. *Depth* 0 *vertices are base objects; depth* 1 *vertices are MetaHyperGraphs of base objects; depth* 2 *vertices are MetaHyperGraphs whose vertices are MetaHyperGraphs, etc. Edge labels are always drawn from the lifted family at the corresponding depth.*

Example 2.23 (Real-life Iterated MetaHyperGraph: hospitals linked by transfers)**.** *Within Hospital* A*, define departmental hypergraphs (patients $\{p_1, p_2, p_4, p_5\}$):*

$$H^{\mathsf{A}}_{\text{Rad}} : \; V = \{p_1, p_2\}, \; E = \{\{p_1, p_2\}\},$$

$$H^{\mathsf{A}}_{\text{Card}} : \; V = \{p_2, p_4\}, \; E = \{\{p_2\}\},$$

$$H^{\mathsf{A}}_{\text{Onc}} : \; V = \{p_2, p_5\}, \; E = \{\{p_2, p_5\}\}.$$

As in the first example, with R_{share}, build the (level-0) MetaHyperGraph

$$\mathsf{M}_{\mathsf{A}} = (V_{\mathsf{A}}, E_{\mathsf{A}}, T_{\mathsf{A}}, Hd_{\mathsf{A}}, \lambda_{\mathsf{A}}), \qquad V_{\mathsf{A}} = \{H^{\mathsf{A}}_{\text{Rad}}, H^{\mathsf{A}}_{\text{Card}}, H^{\mathsf{A}}_{\text{Onc}}\},$$

with a meta-hyperedge e_{A}:

$$T_{\mathsf{A}}(e_{\mathsf{A}}) = \{H^{\mathsf{A}}_{\text{Rad}}, H^{\mathsf{A}}_{\text{Card}}\}, \quad Hd_{\mathsf{A}}(e_{\mathsf{A}}) = \{H^{\mathsf{A}}_{\text{Onc}}\},$$

$$\lambda_{\mathsf{A}}(e_{\mathsf{A}}) = R_{\text{share}},$$

witnessed by patient p_2.
Within Hospital B *(patients $\{p_2, p_3\}$), define*

$$H^{\mathsf{B}}_{\text{Rad}} : \; V = \{p_2, p_3\}, \; E = \{\{p_2\}\}, \qquad H^{\mathsf{B}}_{\text{Card}} : \; V = \{p_3\}, \; E = \{\{p_3\}\}.$$

Its MetaHyperGraph M_{B} has a meta-hyperedge e_{B} with

$$T_{\mathsf{B}}(e_{\mathsf{B}}) = \{H^{\mathsf{B}}_{\text{Card}}\}, \quad Hd_{\mathsf{B}}(e_{\mathsf{B}}) = \{H^{\mathsf{B}}_{\text{Rad}}\},$$

$$\lambda_{\mathsf{B}}(e_{\mathsf{B}}) = R_{\text{share}},$$

witnessed by patient p_3.
Now form the Iterated MetaHyperGraph *at depth* 1 *whose vertices are the level-0 MetaHyperGraphs*

$$\mathscr{V}^{(1)} = \{\mathsf{M}_{\mathsf{A}},\ \mathsf{M}_{\mathsf{B}}\}.$$

Use the lifted relation $R^{\Uparrow}_{\text{share}}$:

$$(S,T) \in R^{\Uparrow}_{\text{share}} \iff \exists M \in S, N \in T,\ \exists A \subseteq V(M), B \subseteq V(N):\ (A,B) \in R_{\text{share}}.$$

Define one meta-hyperedge $E_{\mathsf{A}\to\mathsf{B}}$ with

$$T^{(1)}(E_{\mathsf{A}\to\mathsf{B}}) = \{\mathsf{M}_{\mathsf{A}}\}, \qquad Hd^{(1)}(E_{\mathsf{A}\to\mathsf{B}}) = \{\mathsf{M}_{\mathsf{B}}\},$$

$$\lambda^{(1)}(E_{\mathsf{A}\to\mathsf{B}}) = R^{\Uparrow}_{\text{share}}.$$

Incidence check (explicit witnesses): choose

$$M = \mathsf{M}_{\mathsf{A}},\ N = \mathsf{M}_{\mathsf{B}},$$

$$A = \{H^{\mathsf{A}}_{\text{Onc}}\} \subseteq V(\mathsf{M}_{\mathsf{A}}),\ B = \{H^{\mathsf{B}}_{\text{Rad}}\} \subseteq V(\mathsf{M}_{\mathsf{B}}).$$

Patient p_2 satisfies

$$p_2 \in \{p_2, p_5\} \in E\big(H^{\mathsf{A}}_{\text{Onc}}\big), \qquad p_2 \in \{p_2\} \in E\big(H^{\mathsf{B}}_{\text{Rad}}\big),$$

hence $(A,B) \in R_{\text{share}}$, so $\big(T^{(1)}(E_{\mathsf{A}\to\mathsf{B}}), Hd^{(1)}(E_{\mathsf{A}\to\mathsf{B}})\big) \in R^{\Uparrow}_{\text{share}}$. Thus the depth-1 Iterated MetaHyperGraph correctly captures an inter-hospital linkage induced by shared patient p_2.

Example 2.24 (Depth 1 IMHG: Universities linked by cross-department coauthorship)**. *Base universe (*$t = 0$*).*** *Let $\mathfrak{U}^{(0)}$ be the class of finite (undirected)* department hypergraphs $H = (V,E)$*, where V is a set of author IDs and $E \subseteq \mathscr{P}^*(V)$ lists multi-author papers. Define the admissible set–relation on finite families of such hypergraphs:*

$$(S,T) \in R_{\text{author}} \iff \exists x \text{ (author ID) s.t. } \forall H \in S \cup T\ \exists e \in E(H):\ x \in e.$$

***Two universities as MetaHyperGraphs over** $(\mathfrak{U}^{(0)}, \{R_{\text{author}}\})$**.** Department hypergraphs (author IDs are global):*

$$\begin{aligned} H^{A}_{\text{CS}}:&\ V = \{a_1,a_2\},\ E = \big\{\{a_1,a_2\}\big\},\\ H^{A}_{\text{Math}}:&\ V = \{a_2,a_3\},\ E = \big\{\{a_2,a_3\}\big\},\\ H^{B}_{\text{Bio}}:&\ V = \{b_1,a_2\},\ E = \big\{\{b_1,a_2\}\big\},\\ H^{B}_{\text{Phys}}:&\ V = \{a_2,b_2\},\ E = \big\{\{a_2,b_2\}\big\}. \end{aligned}$$

Build university-level MetaHyperGraphs $\mathsf{M}_A, \mathsf{M}_B$:

$$\mathsf{M}_A = (V_A, E_A, T_A, Hd_A, \lambda_A), \quad V_A = \{H^{A}_{\text{CS}}, H^{A}_{\text{Math}}\},$$

with one meta-hyperedge e_A:

$$T_A(e_A) = \{H^{A}_{\text{CS}}\}, \quad Hd_A(e_A) = \{H^{A}_{\text{Math}}\}, \quad \lambda_A(e_A) = R_{\text{author}},$$

certified by the witness $x = a_2$ since $a_2 \in \{a_1,a_2\} \in E(H^{A}_{\text{CS}})$ and $a_2 \in \{a_2,a_3\} \in E(H^{A}_{\text{Math}})$. Similarly,

$$\mathsf{M}_B = (V_B, E_B, T_B, Hd_B, \lambda_B), \quad V_B = \{H^{B}_{\text{Bio}}, H^{B}_{\text{Phys}}\},$$

with one meta-hyperedge e_B:

$$T_B(e_B) = \{H^{B}_{\text{Bio}}\}, \quad Hd_B(e_B) = \{H^{B}_{\text{Phys}}\}, \quad \lambda_B(e_B) = R_{\text{author}},$$

certified by $x = a_2$ since $a_2 \in \{b_1,a_2\} \in E(H^{B}_{\text{Bio}})$ and $a_2 \in \{a_2,b_2\} \in E(H^{B}_{\text{Phys}})$.
***Depth** $t = 1$ **(IMHG over** $(\mathfrak{U}^{(1)}, \mathscr{R}^{(1)})$**).** Lift the relation:*

$$(S,T) \in R^{\Uparrow}_{\text{author}} \iff \exists M \in S,\ N \in T,\ \exists A \subseteq V(M), B \subseteq V(N):\ (A,B) \in R_{\text{author}}.$$

Define the depth-1 IMHG

$$\mathsf{M}^{(1)} = (V^{(1)}, E^{(1)}, T^{(1)}, Hd^{(1)}, \lambda^{(1)}), \quad V^{(1)} = \{\mathsf{M}_A, \mathsf{M}_B\},$$

with one meta-hyperedge $E_{A\to B}$:

$$T^{(1)}(E_{A\to B}) = \{\mathsf{M}_A\}, \qquad Hd^{(1)}(E_{A\to B}) = \{\mathsf{M}_B\}, \qquad \lambda^{(1)}(E_{A\to B}) = R^{\Uparrow}_{\text{author}}.$$

Incidence check (explicit witnesses). Take $M = \mathsf{M}_A$, $N = \mathsf{M}_B$ and

$$A = \{H^{A}_{\text{Math}}\} \subseteq V(\mathsf{M}_A), \qquad B = \{H^{B}_{\text{Bio}}\} \subseteq V(\mathsf{M}_B).$$

With $x = a_2$, we have $x \in \{a_2,a_3\} \in E(H^{A}_{\text{Math}})$ and $x \in \{b_1,a_2\} \in E(H^{B}_{\text{Bio}})$, hence $(A,B) \in R_{\text{author}}$ and $(T^{(1)}(E_{A\to B}), Hd^{(1)}(E_{A\to B})) \in R^{\Uparrow}_{\text{author}}$, as required.

Example 2.25 (Depth 2 IMHG: Retail stores → chains via shared products)**.** ***Base universe*** *($t=0$). Let $\mathfrak{U}^{(0)}$ be finite* department hypergraphs *$H=(V,E)$ where V is a set of product IDs and $e\in E$ is a* bundle/promotion *(a set of products). Define the admissible set–relation:*

$$(S,T)\in R_{\text{prod}} \iff \exists x \text{ (product) s.t. } \forall H\in S\cup T\ \exists e\in E(H):\ x\in e.$$

Store-level MetaHyperGraphs over $(\mathfrak{U}^{(0)},\{R_{\text{prod}}\})$*. Departments as hypergraphs:*

$$\begin{aligned} H^{S_1}_{\text{Grocery}} &: V=\{\text{Milk},\text{Bread}\},\ E=\{\{\text{Milk},\text{Bread}\}\},\\ H^{S_1}_{\text{Deli}} &: V=\{\text{Bread},\text{Cheese}\},\ E=\{\{\text{Bread},\text{Cheese}\}\},\\ H^{S_2}_{\text{Grocery}} &: V=\{\text{Milk},\text{Eggs}\},\ E=\{\{\text{Milk},\text{Eggs}\}\},\\ H^{S_2}_{\text{Bakery}} &: V=\{\text{Bread},\text{Eggs}\},\ E=\{\{\text{Bread},\text{Eggs}\}\}. \end{aligned}$$

Stores as MetaHyperGraphs:

$$\mathsf{M}_{S_1}=(V_1,E_1,T_1,Hd_1,\lambda_1),\quad V_1=\{H^{S_1}_{\text{Grocery}},H^{S_1}_{\text{Deli}}\},$$

with meta-hyperedge e_1 labeled R_{prod}:

$$T_1(e_1)=\{H^{S_1}_{\text{Grocery}}\},\quad Hd_1(e_1)=\{H^{S_1}_{\text{Deli}}\},\quad \lambda_1(e_1)=R_{\text{prod}},$$

witnessed by $x=$ Bread*. Similarly*

$$\mathsf{M}_{S_2}=(V_2,E_2,T_2,Hd_2,\lambda_2),\quad V_2=\{H^{S_2}_{\text{Grocery}},H^{S_2}_{\text{Bakery}}\},$$

with meta-hyperedge e_2:

$$T_2(e_2)=\{H^{S_2}_{\text{Grocery}}\},\quad Hd_2(e_2)=\{H^{S_2}_{\text{Bakery}}\},\quad \lambda_2(e_2)=R_{\text{prod}},$$

witnessed by $x=$ Eggs*.*
Depth $t=1$ ***(regional IMHG).*** *Lift R_{prod}:*

$$(S,T)\in R^{\Uparrow}_{\text{prod}} \iff \exists M\in S,\ N\in T,\ \exists A\subseteq V(M),B\subseteq V(N):\ (A,B)\in R_{\text{prod}}.$$

Define

$$\mathsf{M}^{(1)}=(V^{(1)},E^{(1)},T^{(1)},Hd^{(1)},\lambda^{(1)}),\quad V^{(1)}=\{\mathsf{M}_{S_1},\mathsf{M}_{S_2}\},$$

with one meta-hyperedge $E_{S_1\to S_2}$ labeled $R^{\Uparrow}_{\text{prod}}$:

$$T^{(1)}(E_{S_1\to S_2})=\{\mathsf{M}_{S_1}\},\qquad Hd^{(1)}(E_{S_1\to S_2})=\{\mathsf{M}_{S_2}\}.$$

Incidence witness at depth 1. Choose $M=\mathsf{M}_{S_1}$, $N=\mathsf{M}_{S_2}$,

$$A=\{H^{S_1}_{\text{Grocery}}\}\subseteq V_1,\qquad B=\{H^{S_2}_{\text{Grocery}}\}\subseteq V_2,$$

and $x=$ Milk*. Since* Milk $\in\{\text{Milk},\text{Bread}\}\in E(H^{S_1}_{\text{Grocery}})$ *and* Milk $\in\{\text{Milk},\text{Eggs}\}\in E(H^{S_2}_{\text{Grocery}})$*, we have $(A,B)\in R_{\text{prod}}$, hence $(T^{(1)}(E_{S_1\to S_2}),Hd^{(1)}(E_{S_1\to S_2}))\in R^{\Uparrow}_{\text{prod}}$.*
Depth $t=2$ ***(chain-level IMHG).*** *Form chain-level MetaHyperGraphs over $(\mathfrak{U}^{(1)},\mathscr{R}^{(1)}=\{R^{\Uparrow}_{\text{prod}}\})$:*

$$\mathsf{C}_{\text{North}}:\ V=\{\mathsf{M}_{S_1}\},\ E=\emptyset;\qquad \mathsf{C}_{\text{South}}:\ V=\{\mathsf{M}_{S_2}\},\ E=\emptyset.$$

Lift again: $\mathscr{R}^{(2)}=\{R^{\Uparrow\uparrow}_{\text{prod}}\}$. Define the depth-2 IMHG

$$\mathsf{M}^{(2)}=(V^{(2)},E^{(2)},T^{(2)},Hd^{(2)},\lambda^{(2)}),\quad V^{(2)}=\{\mathsf{C}_{\text{North}},\mathsf{C}_{\text{South}}\},$$

with one meta-hyperedge $E_{\text{North}\to\text{South}}$:

$$T^{(2)}(E_{\text{North}\to\text{South}})=\{\mathsf{C}_{\text{North}}\},\quad Hd^{(2)}(E_{\text{North}\to\text{South}})=\{\mathsf{C}_{\text{South}}\},\quad \lambda^{(2)}(E_{\text{North}\to\text{South}})=R^{\Uparrow\uparrow}_{\text{prod}}.$$

Incidence witness at depth 2. Take $M=\mathsf{M}_{S_1}\in V(\mathsf{C}_{\text{North}})$ and $N=\mathsf{M}_{S_2}\in V(\mathsf{C}_{\text{South}})$. We already proved $(M,N)\in R^{\Uparrow}_{\text{prod}}$ via $A=\{H^{S_1}_{\text{Grocery}}\}$, $B=\{H^{S_2}_{\text{Grocery}}\}$, $x=$ Milk*. Therefore*

$$\big(T^{(2)}(E_{\text{North}\to\text{South}}),\ Hd^{(2)}(E_{\text{North}\to\text{South}})\big)\in R^{\Uparrow\uparrow}_{\text{prod}},$$

verifying the depth-2 incidence explicitly.

Theorem 2.26 (IMHGs generalize MetaHyperGraphs)**.** *Depth* 0 *IMHGs are exactly MetaHyperGraphs over* $(\mathfrak{U},\mathscr{R})$*.*

Proof. By Definitions 2.20–2.21, taking $t=0$ yields $\mathfrak{U}^{(0)}=\mathfrak{U}$ and $\mathscr{R}^{(0)}=\mathscr{R}$, hence IMHGs of depth 0 are precisely MetaHyperGraphs over $(\mathfrak{U},\mathscr{R})$. □

Definition 2.27 ((Recall) Metagraph and its lift)**.** *Let $\mathfrak{G}$ be a universe and $\mathscr{Q} \subseteq \mathscr{P}(\mathfrak{G}\times\mathfrak{G})$ a family of binary relations. A (labelled)* metagraph *over $(\mathfrak{G},\mathscr{Q})$ is a directed, labelled multigraph $M=(V,E,s,t,\lambda)$ with $V\subseteq\mathfrak{G}$, $s,t: E\to V$, $\lambda: E\to\mathscr{Q}$, and $(s(e),t(e))\in\lambda(e)$ for all e. Its standard lift is: for $R\in\mathscr{Q}$ and metagraphs M_1,M_2,*

$$(M_1,M_2)\in R^{\uparrow} \iff \exists x\in V(M_1),\, y\in V(M_2):\ (x,y)\in R.$$

Define iterated metagraph universes by $\mathfrak{G}^{(0)} := \mathfrak{G}$, $\mathscr{Q}^{(0)} := \mathscr{Q}$, and

$$\mathfrak{G}^{(t+1)} := \{\textit{finite metagraphs over } (\mathfrak{G}^{(t)},\mathscr{Q}^{(t)})\},$$

$$\mathscr{Q}^{(t+1)} := \big(\mathscr{Q}^{(t)}\big)^{\uparrow}.$$

Definition 2.28 (Singleton embedding of binary relations)**.** *Embed $\mathscr{Q}$ into set–relations by*

$$\iota:\ \mathscr{Q}\longrightarrow \mathscr{P}\Big(\mathscr{P}_{\text{fin}}(\mathfrak{G})\times\mathscr{P}_{\text{fin}}(\mathfrak{G})\Big),\qquad \iota(R) := \big\{\,(\{x\},\{y\})\mid (x,y)\in R\,\big\}.$$

Lemma 2.29 (Lift commutes with ι on singletons)**.** *Let $t\geq 0$. View ι levelwise, i.e. as*

$$\iota:\mathscr{Q}^{(t)}\to\mathscr{P}(\mathscr{P}_{\text{fin}}(\mathfrak{G}^{(t)})\times\mathscr{P}_{\text{fin}}(\mathfrak{G}^{(t)}))$$

. Then

$$\big(\iota(R)\big)^{\Uparrow}\cap\big\{(\{M\},\{N\})\big\}\ =\ \iota\big(R^{\uparrow}\big)$$

for all $R\in\mathscr{Q}^{(t)}$.

Proof. By Definition 2.19, for $(\{M\},\{N\})$ we have

$$\begin{aligned}
&(\{M\},\{N\})\in\big(\iota(R)\big)^{\Uparrow}\\
&\iff \exists A\subseteq V(M),\,B\subseteq V(N):\ (A,B)\in\iota(R)\\
&\iff \exists x\in V(M),\,y\in V(N):\ (\{x\},\{y\})\in\iota(R)\\
&\iff \exists x\in V(M),\,y\in V(N):\ (x,y)\in R\\
&\iff (M,N)\in R^{\uparrow}.
\end{aligned}$$

This is exactly $(\{M\},\{N\})\in\iota(R^{\uparrow})$. ☐

Theorem 2.30 (IMHGs generalize Iterated MetaGraphs)**.** *Fix $t\geq 0$. Consider IMHGs over $\big(\mathfrak{G}^{(t)},\,\iota(\mathscr{Q}^{(t)})\big)$ and restrict to the full subcategory $\mathbf{IMHG}_1^{(t)}$ in which every hyperedge e satisfies $|T^{(t)}(e)|=|Hd^{(t)}(e)|=1$. Then $\mathbf{IMHG}_1^{(t)}$ is canonically equivalent to the category of iterated metagraphs of depth t over $\big(\mathfrak{G}^{(t)},\mathscr{Q}^{(t)}\big)$.*

Proof. Define functors F_t and G_t (mutually inverse on objects and morphisms).
(F_t : metagraph → IMHG) Given a depth-t metagraph

$$M^{(t)}=(V,E,s,t,\lambda)\quad\text{over } (\mathfrak{G}^{(t)},\mathscr{Q}^{(t)}),$$

set $\mathsf{M}^{(t)} := F_t(M^{(t)})$ with the same vertex set V, edge set E, and for each $e\in E$ put

$$T^{(t)}(e):=\{s(e)\},\qquad Hd^{(t)}(e):=\{t(e)\},\qquad \lambda^{(t)}(e):=\iota\big(\lambda(e)\big).$$

Incidence holds because $(s(e),t(e))\in\lambda(e)$ iff $(\{s(e)\},\{t(e)\})\in\iota(\lambda(e))$.
(G_t : IMHG → metagraph) Given a depth-t IMHG in $\mathbf{IMHG}_1^{(t)}$,

$$\mathsf{M}^{(t)}=(V,E,T,Hd,\lambda)\quad\text{over } (\mathfrak{G}^{(t)},\iota(\mathscr{Q}^{(t)})),$$

define $M^{(t)} := G_t(\mathsf{M}^{(t)})$ with the same vertices V, edges E, and for each e take $s(e)$, $t(e)$ to be the unique elements of $T(e)$, $Hd(e)$, and set $\lambda(e)\in\mathscr{Q}^{(t)}$ uniquely so that $\lambda^{(t)}(e)=\iota\big(\lambda(e)\big)$. Incidence is equivalent by the definition of ι.
Compatibility with depth-t labels (which are lifts of depth-$(t-1)$ labels) follows from Lemma 2.29: the lifted label families correspond under ι when restricting to singleton tails/heads. Functoriality on morphisms (label-preserving incidence-commuting maps) is inherited verbatim. Clearly $G_t\circ F_t=\text{id}$ and $F_t\circ G_t=\text{id}$ on the stated subcategory, yielding the claimed equivalence. ☐

2.4. MetaSuperHyperGraph(SuperHyperGraph of SuperHyperGraph)

A MetaSuperHyperGraph is a superhypergraph whose vertices are themselves superhypergraphs, with superhyperedges describing relations between these higher-order structures.

Notation 2.31. *For a set X, write $\mathscr{P}_{\mathrm{fin}}(X)$ for the family of all finite subsets of X and $\mathscr{P}^*_{\mathrm{fin}}(X) := \mathscr{P}_{\mathrm{fin}}(X) \setminus \{\emptyset\}$. For a function f and a set A, put $f[A] := \{f(a) : a \in A\}$.*

Definition 2.32 (Iterated singleton embedding)**.** *Let U be any set. Define $j_0 : U \to U$ by $j_0(x) := x$ and inductively $j_{k+1} : U \to \mathscr{P}(\mathscr{P}^k(U))$ by $j_{k+1}(x) := \{ j_k(x) \}$. Thus $j_n : U \hookrightarrow \mathscr{P}^n(U)$ is the canonical depth-n embedding.*

Definition 2.33 (Directed n-SuperHyperGraph)**.** *Fix a base set Ω and $n \in \mathbb{N}_0$. A* directed n-SuperHyperGraph *is a tuple*

$$\mathscr{S} = (V, E, T, Hd) \quad \textit{with} \quad V \subseteq \mathscr{P}^n(\Omega),\ \ T, Hd : E \to \mathscr{P}_{\mathrm{fin}}(V),$$

such that $T(e) \cup Hd(e) \neq \emptyset$ for all $e \in E$. (Undirected variants arise by requiring $T(e) = Hd(e)$ for all e.)

Definition 2.34 (Universe of n-SuperHyperGraphs)**.** *Let $\mathfrak{S}_n$
denote the class of all finite directed n-SuperHyperGraphs over arbitrary base sets. For*

$$x \in \mathscr{P}^n(\Omega)$$

, the unit n-SuperHyperGraph *at x is*

$$\mathsf{U}_n(x) \ := \ \big(\{x\}, \emptyset, \_, \_\big) \ \in \ \mathfrak{S}_n.$$

Definition 2.35 (MetaSuperHyperGraph over $(\mathfrak{S}_n, \mathscr{R})$)**.** *Let $\mathfrak{S}_n$ be as above and let*

$$\mathscr{R} \ \subseteq \ \mathscr{P}\Big(\ \mathscr{P}_{\mathrm{fin}}(\mathfrak{S}_n) \times \mathscr{P}_{\mathrm{fin}}(\mathfrak{S}_n) \ \Big)$$

be any nonempty family of admissible set–relations on n-SuperHyperGraphs. *A* MetaSuperHyperGraph (MSHG) *over $(\mathfrak{S}_n, \mathscr{R})$ is a labelled directed hypergraph*

$$\mathsf{M} = (V, E, T, Hd, \lambda)$$

with $V \subseteq \mathfrak{S}_n$, $T, Hd : E \to \mathscr{P}_{\mathrm{fin}}(V)$, and $\lambda : E \to \mathscr{R}$, satisfying the incidence constraint

$$\forall e \in E : \quad \big(T(e), Hd(e)\big) \ \in \ \lambda(e).$$

Vertices of M are (finite) n-SuperHyperGraphs; hyperedges relate finite families of such vertices.

Example 2.36 (Real-life MetaSuperHyperGraph: multi-department clinical cohorts)**.** *Let the national patient-ID universe be $\Omega = \{u_1, u_2, u_3, u_4, u_5\}$ and consider depth-*1 *SuperHyperGraphs (vertices are subsets of Ω).*
Department A (oncology) as a 1*-SuperHyperGraph:*

$$H_A = (V_A, E_A, T_A, Hd_A), \quad V_A = \{\{u_1, u_2\},\ \{u_2, u_3\}\}, \quad E_A = \{e_A\},$$

$$T_A(e_A) = \big\{\{u_1, u_2\}\big\}, \qquad Hd_A(e_A) = \big\{\{u_2, u_3\}\big\}.$$

Department B (radiology):

$$H_B = (V_B, E_B, T_B, Hd_B), \quad V_B = \{\{u_2, u_4\}\}, \quad E_B = \emptyset.$$

Department C (cardiology):

$$H_C = (V_C, E_C, T_C, Hd_C), \quad V_C = \{\{u_1, u_2, u_5\}\}, \quad E_C = \emptyset.$$

Define the admissible set–relation on 1*-SuperHyperGraphs*

$$R_{\mathrm{shareV}} : \ (S, T) \in R_{\mathrm{shareV}} \iff \exists x \in \Omega \text{ s.t. } \forall H \in S \cup T \ \exists A_H \in V(H) \text{ with } x \in A_H.$$

Form the MetaSuperHyperGraph $\mathsf{M} = (V, E, T, Hd, \lambda)$ over the universe $V = \{H_A, H_B, H_C\}$ with one meta-hyperedge $e^\star$:

$$T(e^\star) = \{H_A, H_B\}, \qquad Hd(e^\star) = \{H_C\}, \qquad \lambda(e^\star) = R_{\mathrm{shareV}}.$$

Incidence check (explicit witness): pick $x = u_2$. Then $u_2 \in \{u_1, u_2\} \in V_A$, $u_2 \in \{u_2, u_4\} \in V_B$, and $u_2 \in \{u_1, u_2, u_5\} \in V_C$, so $(T(e^\star), Hd(e^\star)) \in R_{\mathrm{shareV}}$ as required.

Example 2.37 (MetaSuperHyperGraph for urban transit agencies sharing a transfer hub (n=1)). ***Base (depth-1) SuperHyperGraphs.*** *Let the stop universe be* $\Omega = \{s_1, s_2, s_3, s_4\}$ *(station/stop IDs). A 1-SuperHyperGraph has vertices that are subsets of* Ω*.*
Rail agency $H_{\mathrm{Rail}} = (V_{\mathrm{R}}, E_{\mathrm{R}}, T_{\mathrm{R}}, Hd_{\mathrm{R}})$:

$$V_{\mathrm{R}} = \big\{\{s_1,s_2\},\ \{s_2,s_3\}\big\}, \quad E_{\mathrm{R}} = \{e_{\mathrm{R}}\},\ T_{\mathrm{R}}(e_{\mathrm{R}}) = \big\{\{s_1,s_2\}\big\},\ Hd_{\mathrm{R}}(e_{\mathrm{R}}) = \big\{\{s_2,s_3\}\big\}.$$

Bus agency $H_{\mathrm{Bus}} = (V_{\mathrm{B}}, E_{\mathrm{B}}, T_{\mathrm{B}}, Hd_{\mathrm{B}})$:

$$V_{\mathrm{B}} = \big\{\{s_2,s_4\}\big\}, \quad E_{\mathrm{B}} = \emptyset.$$

Ferry agency $H_{\mathrm{Ferry}} = (V_{\mathrm{F}}, E_{\mathrm{F}}, T_{\mathrm{F}}, Hd_{\mathrm{F}})$:

$$V_{\mathrm{F}} = \big\{\{s_2\}\big\}, \quad E_{\mathrm{F}} = \emptyset.$$

Each $H_\bullet$ *is a finite directed 1-SuperHyperGraph since* $V_\bullet \subseteq \mathscr{P}(\Omega)$ *and* $T_\bullet, Hd_\bullet$ *(when defined) map edges into* $\mathscr{P}_{\mathrm{fin}}(V_\bullet)$*.*
Admissible set–relation on $\mathfrak{S}_1$***.*** *Define* R_{hub} *on finite families of 1-SuperHyperGraphs by*

$$(S,T) \in R_{\mathrm{hub}} \iff \exists x \in \Omega \text{ such that } \forall H \in S \cup T\ \exists A_H \in V(H) \text{ with } x \in A_H.$$

Intuition: every agency in the tail and head "touches" the same transfer hub x.
MetaSuperHyperGraph over $(\mathfrak{S}_1, \{R_{\mathrm{hub}}\})$***.*** *Let*

$$V = \{H_{\mathrm{Rail}}, H_{\mathrm{Bus}}, H_{\mathrm{Ferry}}\}, \qquad E = \{e^\star\}.$$

Define the meta-hyperedge $e^\star$ *by*

$$T(e^\star) = \{H_{\mathrm{Rail}}, H_{\mathrm{Bus}}\}, \qquad Hd(e^\star) = \{H_{\mathrm{Ferry}}\}, \qquad \lambda(e^\star) = R_{\mathrm{hub}}.$$

Incidence verification (explicit witness). Choose $x = s_2$*. Then*

$$s_2 \in \{s_1,s_2\} \in V_{\mathrm{R}}, \qquad s_2 \in \{s_2,s_4\} \in V_{\mathrm{B}}, \qquad s_2 \in \{s_2\} \in V_{\mathrm{F}}.$$

Hence $\big(T(e^\star), Hd(e^\star)\big) \in R_{\mathrm{hub}}$ *and the MSHG incidence constraint holds.*

Example 2.38 (MetaSuperHyperGraph for online courses sharing a student across group-of-groups (n=2)). ***Base (depth-2) SuperHyperGraphs.*** *Let the student universe be* $\Omega = \{u_1,u_2,u_3,u_4,u_5\}$*. A 2-SuperHyperGraph has vertices* $V \subseteq \mathscr{P}^2(\Omega)$*, i.e., each vertex is a* set of study groups *(each group is a subset of* Ω*).*
Course ML $H_{\mathrm{ML}} = (V_{\mathrm{ML}}, E_{\mathrm{ML}}, T_{\mathrm{ML}}, Hd_{\mathrm{ML}})$:

$$A_1 = \big\{\{u_1,u_2\},\{u_2,u_3\}\big\}, \quad A_2 = \big\{\{u_3,u_4\}\big\},$$

$$V_{\mathrm{ML}} = \{A_1, A_2\}, \quad E_{\mathrm{ML}} = \{e_{\mathrm{ML}}\},\ T_{\mathrm{ML}}(e_{\mathrm{ML}}) = \{A_1\},\ Hd_{\mathrm{ML}}(e_{\mathrm{ML}}) = \{A_2\}.$$

Course DS $H_{\mathrm{DS}} = (V_{\mathrm{DS}}, E_{\mathrm{DS}}, T_{\mathrm{DS}}, Hd_{\mathrm{DS}})$:

$$B_1 = \big\{\{u_2,u_5\}\big\}, \quad B_2 = \big\{\{u_1,u_2,u_5\}\big\}, \quad V_{\mathrm{DS}} = \{B_1,B_2\}, \quad E_{\mathrm{DS}} = \emptyset.$$

Course AI $H_{\mathrm{AI}} = (V_{\mathrm{AI}}, E_{\mathrm{AI}}, T_{\mathrm{AI}}, Hd_{\mathrm{AI}})$:

$$C_1 = \big\{\{u_2\},\{u_4\}\big\}, \quad V_{\mathrm{AI}} = \{C_1\}, \quad E_{\mathrm{AI}} = \emptyset.$$

Each $H_\bullet$ *is a finite directed 2-SuperHyperGraph, since* $A_i, B_j, C_1 \in \mathscr{P}^2(\Omega)$ *and the edge maps (where present) take values in* $\mathscr{P}_{\mathrm{fin}}(V_\bullet)$*.*
Admissible set–relation on $\mathfrak{S}_2$***.*** *Define* R_{student} *on finite families of 2-SuperHyperGraphs by*

$$(S,T) \in R_{\mathrm{student}} \iff \exists x \in \Omega \text{ such that } \forall H \in S \cup T\ \exists U_H \in V(H)\ \exists G_H \in U_H \text{ with } x \in G_H.$$

Intuition: the same student x occurs in at least one underlying group inside every course in the tail and head.
MetaSuperHyperGraph over $(\mathfrak{S}_2, \{R_{\mathrm{student}}\})$***.*** *Let*

$$V = \{H_{\mathrm{ML}}, H_{\mathrm{DS}}, H_{\mathrm{AI}}\}, \qquad E = \{e^\diamond\}.$$

Define the meta-hyperedge $e^\diamond$ *by*

$$T(e^\diamond) = \{H_{\mathrm{ML}}\}, \qquad Hd(e^\diamond) = \{H_{\mathrm{DS}}, H_{\mathrm{AI}}\}, \qquad \lambda(e^\diamond) = R_{\mathrm{student}}.$$

Incidence verification (explicit witness). Choose $x = u_2$*. Then we can select*

$$U_{H_{\mathrm{ML}}} = A_1 \in V_{\mathrm{ML}},\ \ G_{H_{\mathrm{ML}}} = \{u_2,u_3\} \in A_1,\ \ u_2 \in G_{H_{\mathrm{ML}}};$$

$$U_{H_{\mathrm{DS}}} = B_1 \in V_{\mathrm{DS}},\ \ G_{H_{\mathrm{DS}}} = \{u_2,u_5\} \in B_1,\ \ u_2 \in G_{H_{\mathrm{DS}}};$$

$$U_{H_{\mathrm{AI}}} = C_1 \in V_{\mathrm{AI}},\ \ G_{H_{\mathrm{AI}}} = \{u_2\} \in C_1,\ \ u_2 \in G_{H_{\mathrm{AI}}}.$$

Thus $\big(T(e^\diamond), Hd(e^\diamond)\big) \in R_{\mathrm{student}}$*, so the MSHG incidence constraint is satisfied.*

Definition 2.39 (Singleton wrapping of relations)**.** *For* $n \geq 0$ *define*

$$\iota_n : \mathscr{Q} \longrightarrow \mathscr{P}\Big(\mathscr{P}_{\mathrm{fin}}(\mathfrak{S}_n) \times \mathscr{P}_{\mathrm{fin}}(\mathfrak{S}_n)\Big), \quad \iota_n(R) := \Big\{ \big(\mathsf{U}_n[S], \mathsf{U}_n[T]\big) \ :\ (S,T) \in R \Big\}.$$

Theorem 2.40 (MSHG ⊃ MetaHyperGraph)**.** *Fix $n \geq 0$. Consider the full subcategory $\mathbf{MSHG}_n^{\mathrm{unit}}$ of MSHGs over $(\mathfrak{S}_n, \iota_n(\mathscr{Q}))$ whose vertex sets are contained in $\mathsf{U}_n[\mathfrak{U}]$. Then $\mathbf{MSHG}_n^{\mathrm{unit}}$ is canonically equivalent to the category of MetaHyperGraphs over $(\mathfrak{U}, \mathscr{Q})$.*

Proof. Define mutually inverse functors on objects and morphisms.
(F : MHG → MSHG) Given $\mathsf{H} = (W, F, S, H, \mu)$, put

$$V := \mathsf{U}_n[W], \quad E := F, \quad T(e) := \mathsf{U}_n[S(e)], \quad Hd(e) := \mathsf{U}_n[H(e)], \quad \lambda(e) := \iota_n\big(\mu(e)\big).$$

Then $(T(e), Hd(e)) \in \iota_n(\mu(e))$ iff $(S(e), H(e)) \in \mu(e)$ by Definition 2.39, so $\mathsf{M} := F(\mathsf{H})$ is an MSHG in $\mathbf{MSHG}_n^{\mathrm{unit}}$.
(G : MSHG → MHG) Conversely, let $\mathsf{M} = (V, E, T, Hd, \lambda)$ lie in $\mathbf{MSHG}_n^{\mathrm{unit}}$. Write $W := \{x \in \mathfrak{U} \mid \mathsf{U}_n(x) \in V\}$ and define

$$S(e) := \{x \in W \mid \mathsf{U}_n(x) \in T(e)\}, \qquad H(e) := \{y \in W \mid \mathsf{U}_n(y) \in Hd(e)\}.$$

For each e choose $\mu(e) \in \mathscr{Q}$ with $\lambda(e) = \iota_n(\mu(e))$ (uniquely determined by Definition 2.39). Then $(S(e), H(e)) \in \mu(e)$ iff $(T(e), Hd(e)) \in \lambda(e)$, so $\mathsf{H} := G(\mathsf{M}) = (W, E, S, H, \mu)$ is an MHG. It is immediate that $G \circ F = \mathrm{id}$ and $F \circ G = \mathrm{id}$. □

Theorem 2.41 (MSHG ⊃ SuperHyperGraph)**.** *Fix a base set Ω and $n \geq 0$. Let $\mathbf{U}$ denote the* universal *set–relation on $\mathfrak{S}_n$:*

$$\mathbf{U} := \mathscr{P}_{\mathrm{fin}}(\mathfrak{S}_n) \times \mathscr{P}_{\mathrm{fin}}(\mathfrak{S}_n), \qquad \mathscr{R} := \{\mathbf{U}\}.$$

The assignment

$$\Phi : \{\textit{directed } n\textit{-SuperHyperGraphs on } \Omega\} \longrightarrow \{\textit{MSHGs over } (\mathfrak{S}_n, \mathscr{R})\}$$

given by

$$\Phi(V, E, T, Hd) := \big(\, \mathsf{U}_n[V],\ E,\ \mathsf{U}_n[T(\cdot)],\ \mathsf{U}_n[Hd(\cdot)],\ \lambda \equiv \mathbf{U} \,\big)$$

is a bijection on isomorphism classes. In particular, every n-SuperHyperGraph is a specialization of an MSHG.

Proof. For any n-SuperHyperGraph (V, E, T, Hd), the image has vertex set $\mathsf{U}_n[V]$ and for each $e \in E$,

$$\big(T'(e), Hd'(e)\big) = \big(\mathsf{U}_n[T(e)], \mathsf{U}_n[Hd(e)]\big) \in \mathbf{U}$$

by definition of $\mathbf{U}$, hence $\Phi(V, E, T, Hd)$ is an MSHG. Conversely, given an MSHG $(V', E', T', Hd', \lambda \equiv \mathbf{U})$ whose vertices are all of the form $\mathsf{U}_n(x)$ with $x \in \mathscr{P}^n(\Omega)$, define

$$V := \{x : \mathsf{U}_n(x) \in V'\}, \qquad T(e) := \{x : \mathsf{U}_n(x) \in T'(e)\}, \qquad Hd(e) := \{x : \mathsf{U}_n(x) \in Hd'(e)\}.$$

Then (V, E', T, Hd) is a directed n-SuperHyperGraph on Ω. Isomorphisms correspond by the obvious componentwise identification, proving the bijection on isomorphism classes. □

2.5. Iterated MetaSuperHyperGraph (SuperHyperGraph of SuperHyperGraph of ... of SuperHyperGraph)

An Iterated MetaSuperHyperGraph is a superhypergraph whose vertices are meta-superhypergraphs, recursively extending superhypergraph-of-superhypergraph structure across multiple hierarchical levels.

Definition 2.42 (Iterated universes for MetaSuperHyperGraphs)**.** *Fix $n \in \mathbb{N}_0$. Let $\mathfrak{S}_n$ be the class of all finite directed n-SuperHyperGraphs (as used previously) and let $\mathscr{R} \subseteq \mathscr{P}\big(\mathscr{P}_{\mathrm{fin}}(\mathfrak{S}_n) \times \mathscr{P}_{\mathrm{fin}}(\mathfrak{S}_n)\big)$ be a nonempty family of admissible set–relations (as in the MetaSuperHyperGraph section). Define, recursively for $t \in \mathbb{N}_0$,*

$$\mathfrak{S}_n^{(0)} := \mathfrak{S}_n, \qquad \mathscr{R}^{(0)} := \mathscr{R},$$

$$\mathfrak{S}_n^{(t+1)} := \Big\{ \textit{finite MetaSuperHyperGraphs over } \big(\mathfrak{S}_n^{(t)}, \mathscr{R}^{(t)}\big) \Big\},$$

$$\mathscr{R}^{(t+1)} := \big(\mathscr{R}^{(t)}\big)^{\Uparrow}, \quad \textit{where } (S, T) \in R^{\Uparrow} \iff$$

$$\exists M \in S, N \in T,\ \exists A \in \mathscr{P}_{\mathrm{fin}}\big(V(M)\big), B \in \mathscr{P}_{\mathrm{fin}}\big(V(N)\big) : \ (A, B) \in R.$$

(Here $V(M)$ denotes the vertex set of M, and $\Uparrow$ is the same lifting operator used earlier.)

Definition 2.43 (Iterated MetaSuperHyperGraph (IMSHG) of depth t)**.** *For $t \in \mathbb{N}_0$, an* Iterated MetaSuperHyperGraph of depth t *is a labelled directed hypergraph*

$$\mathsf{M}^{(t)} = (V^{(t)}, E^{(t)}, T^{(t)}, Hd^{(t)}, \lambda^{(t)})$$

such that

$$V^{(t)} \subseteq \mathfrak{S}_n^{(t)}, \qquad T^{(t)}, Hd^{(t)} : E^{(t)} \to \mathscr{P}_{\mathrm{fin}}\big(V^{(t)}\big),$$

$$\lambda^{(t)} : E^{(t)} \to \mathscr{R}^{(t)},$$

and the incidence constraint holds:

$$\forall e \in E^{(t)} : \ \big(T^{(t)}(e), Hd^{(t)}(e)\big) \in \lambda^{(t)}(e).$$

Example 2.44 (Real-life Iterated MetaSuperHyperGraph: hospital networks via shared cohorts)**.** *Extend the patient-ID universe to* $\Omega' = \{u_1, u_2, u_3, u_4, u_5, u_6\}$.
Hospital A *uses the three* 1*-SuperHyperGraphs above and forms the MetaSuperHyperGraph*

$$\mathsf{M}_{\mathsf{A}} = (V_{\mathsf{A}}, E_{\mathsf{A}}, T_{\mathsf{A}}, Hd_{\mathsf{A}}, \lambda_{\mathsf{A}}),$$

$$V_{\mathsf{A}} = \{H_A, H_B, H_C\},$$

with the meta-hyperedge e_{A} *already verified by* $x = u_2$*:*

$$T_{\mathsf{A}}(e_{\mathsf{A}}) = \{H_A, H_B\}, \quad Hd_{\mathsf{A}}(e_{\mathsf{A}}) = \{H_C\},$$

$$\lambda_{\mathsf{A}}(e_{\mathsf{A}}) = R_{\text{shareV}}.$$

Hospital B *builds two* 1*-SuperHyperGraphs using* Ω'*:*

$$H_D = (V_D, E_D, T_D, Hd_D), \quad V_D = \{\{u_2, u_6\}\},\ E_D = \emptyset;$$

$$H_E = (V_E, E_E, T_E, Hd_E), \quad V_E = \{\{u_2\}\},\ E_E = \emptyset.$$

Its MetaSuperHyperGraph is

$$\mathsf{M}_{\mathsf{B}} = (V_{\mathsf{B}}, E_{\mathsf{B}}, T_{\mathsf{B}}, Hd_{\mathsf{B}}, \lambda_{\mathsf{B}}),$$

$$V_{\mathsf{B}} = \{H_D, H_E\},$$

with one meta-hyperedge e_{B} *witnessed by* $x = u_2$*:*

$$T_{\mathsf{B}}(e_{\mathsf{B}}) = \{H_E\}, \quad Hd_{\mathsf{B}}(e_{\mathsf{B}}) = \{H_D\},$$

$$\lambda_{\mathsf{B}}(e_{\mathsf{B}}) = R_{\text{shareV}}.$$

*Lift the relation to MetaSuperHyperGraphs (depth-*1 *lifting):*

$$(S,T) \in R^{\Uparrow}_{\text{shareV}} \iff \exists M \in S, N \in T,\ \exists A \subseteq V(M), B \subseteq V(N):\ (A,B) \in R_{\text{shareV}}.$$

*Construct the depth-*1 *Iterated MetaSuperHyperGraph*

$$\mathbf{M}^{(1)} = \big(V^{(1)}, E^{(1)}, T^{(1)}, Hd^{(1)}, \lambda^{(1)}\big),$$

$$V^{(1)} = \{\mathsf{M}_{\mathsf{A}}, \mathsf{M}_{\mathsf{B}}\},$$

with one meta-hyperedge $E_{\mathsf{A}\to\mathsf{B}}$*:*

$$T^{(1)}(E_{\mathsf{A}\to\mathsf{B}}) = \{\mathsf{M}_{\mathsf{A}}\},$$

$$Hd^{(1)}(E_{\mathsf{A}\to\mathsf{B}}) = \{\mathsf{M}_{\mathsf{B}}\},$$

$$\lambda^{(1)}(E_{\mathsf{A}\to\mathsf{B}}) = R^{\Uparrow}_{\text{shareV}}.$$

Incidence verification (explicit witnesses): choose $M = \mathsf{M}_{\mathsf{A}}$*,* $N = \mathsf{M}_{\mathsf{B}}$*,*

$$A = \{H_A\} \subseteq V_{\mathsf{A}}, \qquad B = \{H_D\} \subseteq V_{\mathsf{B}}.$$

Take $x = u_2$*. Then* $u_2 \in \{u_1, u_2\} \in V_A$ *and* $u_2 \in \{u_2, u_6\} \in V_D$*, so* $(A,B) \in R_{\text{shareV}}$*, hence*

$$\big(T^{(1)}(E_{\mathsf{A}\to\mathsf{B}}), Hd^{(1)}(E_{\mathsf{A}\to\mathsf{B}})\big) \in R^{\Uparrow}_{\text{shareV}}.$$

Therefore $\mathbf{M}^{(1)}$ *is a valid depth-*1 *Iterated MetaSuperHyperGraph linking hospitals by shared patient cohorts.*

Example 2.45 (Depth 1 IMSHG: Open-source foundations linked by shared contributors (n=1))**.** ***Level*** $t = 0$***:*** **1*-SuperHyperGraphs (projects).*** *Let the contributor universe be* $\Omega = \{d_1, d_2, d_3, d_4, d_5\}$*. A* 1*-SuperHyperGraph has* $V \subseteq \mathscr{P}(\Omega)$*, i.e., each vertex is a* contributor group.
Projects (each a finite directed 1-SuperHyperGraph):

$$\begin{aligned} H_{\text{kernel}} &= (V_K, E_K, T_K, Hd_K), \quad V_K = \big\{\{d_1, d_2\}, \{d_2, d_3\}\big\}, \\ &\quad E_K = \{e_K\},\ T_K(e_K) = \big\{\{d_1, d_2\}\big\},\ Hd_K(e_K) = \big\{\{d_2, d_3\}\big\}; \\ H_{\text{tools}} &= (V_T, E_T, T_T, Hd_T), \quad V_T = \big\{\{d_2, d_4\}\big\},\ E_T = \emptyset; \\ H_{\text{web}} &= (V_W, E_W, T_W, Hd_W), \quad V_W = \big\{\{d_3, d_5\}, \{d_2\}\big\},\ E_W = \emptyset; \\ H_{\text{cli}} &= (V_C, E_C, T_C, Hd_C), \quad V_C = \big\{\{d_2, d_5\}\big\},\ E_C = \emptyset. \end{aligned}$$

Admissible set–relation on $\mathfrak{S}_1$**.** *Define* R_{contrib} *on finite families of* 1*-SuperHyperGraphs by*

$$(S,T) \in R_{\text{contrib}} \iff \exists x \in \Omega \text{ s.t. } \forall H \in S \cup T\ \exists A_H \in V(H) \text{ with } x \in A_H.$$

Intuition: all selected projects share at least one common contributor x somewhere *in their group vertices.*

Level $t=0$***: MetaSuperHyperGraphs (foundations) over*** $(\mathfrak{S}_1,\{R_{\text{contrib}}\})$***.***

$$\mathsf{M}_{\text{F1}}=(V_1,E_1,T_1,Hd_1,\lambda_1),\quad V_1=\{H_{\text{kernel}},H_{\text{tools}}\},$$

with one meta-hyperedge e_1*:*

$$T_1(e_1)=\{H_{\text{kernel}}\},\quad Hd_1(e_1)=\{H_{\text{tools}}\},\quad \lambda_1(e_1)=R_{\text{contrib}}.$$

Witness for incidence: *take* $x=d_2$*; then* $d_2\in\{d_1,d_2\}\in V_K$ *and* $d_2\in\{d_2,d_4\}\in V_T$*.*
Similarly,

$$\mathsf{M}_{\text{F2}}=(V_2,E_2,T_2,Hd_2,\lambda_2),\quad V_2=\{H_{\text{web}},H_{\text{cli}}\},$$

with meta-hyperedge e_2*:*

$$T_2(e_2)=\{H_{\text{web}}\},\quad Hd_2(e_2)=\{H_{\text{cli}}\},\quad \lambda_2(e_2)=R_{\text{contrib}}.$$

Witness: $x=d_2$*; indeed* $d_2\in\{d_2\}\in V_W$ *and* $d_2\in\{d_2,d_5\}\in V_C$*.*
Level $t=1$***: IMSHG over*** $(\mathfrak{S}_1^{(1)},\mathscr{R}^{(1)}=\{R^{\Uparrow}_{\text{contrib}}\})$***.*** *Define*

$$\mathsf{M}^{(1)}=(V^{(1)},E^{(1)},T^{(1)},Hd^{(1)},\lambda^{(1)}),\quad V^{(1)}=\{\mathsf{M}_{\text{F1}},\mathsf{M}_{\text{F2}}\}.$$

Introduce one meta-hyperedge $E_{\text{F1}\to\text{F2}}$ *with*

$$T^{(1)}(E_{\text{F1}\to\text{F2}})=\{\mathsf{M}_{\text{F1}}\},\quad Hd^{(1)}(E_{\text{F1}\to\text{F2}})=\{\mathsf{M}_{\text{F2}}\},\quad \lambda^{(1)}(E_{\text{F1}\to\text{F2}})=R^{\Uparrow}_{\text{contrib}}.$$

Incidence verification (explicit witnesses): *Choose* $M=\mathsf{M}_{\text{F1}}$*,* $N=\mathsf{M}_{\text{F2}}$ *and*

$$A=\{H_{\text{tools}}\}\subseteq V_1,\qquad B=\{H_{\text{cli}}\}\subseteq V_2.$$

With $x=d_2$*, we have* $d_2\in\{d_2,d_4\}\in V_T$ *and* $d_2\in\{d_2,d_5\}\in V_C$*, hence* $(A,B)\in R_{\text{contrib}}$ *and so*

$$\big(T^{(1)}(E_{\text{F1}\to\text{F2}}),Hd^{(1)}(E_{\text{F1}\to\text{F2}})\big)\in R^{\Uparrow}_{\text{contrib}}.$$

Therefore $\mathsf{M}^{(1)}$ *is a valid depth-*1 *IMSHG linking foundations via a shared contributor.*

Example 2.46 (Depth 2 IMSHG: Emergency response regions linked by shared responders (n=2))**.** ***Level*** $t=0$***:*** **2*-SuperHyperGraphs (city agencies).*** *Let the responder universe be* $\Omega=\{r_1,r_2,r_3,r_4,r_5,r_6\}$*. A* 2*-SuperHyperGraph has* $V\subseteq\mathscr{P}^2(\Omega)$*: each vertex is a* set of unit rosters *(each roster is a subset of* Ω*).*
City A agencies:

$$\begin{aligned}H^A_{\text{Fire}}=(V^A_F,E^A_F,T^A_F,Hd^A_F),\quad & U_1=\big\{\{r_1,r_2\},\{r_2\}\big\},\ U_2=\big\{\{r_2,r_3\}\big\},\\ & V^A_F=\{U_1,U_2\},\ E^A_F=\{e_F\},\ T^A_F(e_F)=\{U_1\},\ Hd^A_F(e_F)=\{U_2\};\\ H^A_{\text{Med}}=(V^A_M,E^A_M,T^A_M,Hd^A_M),\quad & V^A_M=\big\{M_1\big\},\ M_1=\big\{\{r_2,r_4\}\big\},\ E^A_M=\emptyset.\end{aligned}$$

City B agencies:

$$\begin{aligned}H^B_{\text{Police}}=(V^B_P,E^B_P,T^B_P,Hd^B_P),\quad & V^B_P=\big\{P_1\big\},\ P_1=\big\{\{r_2,r_6\}\big\},\ E^B_P=\emptyset;\\ H^B_{\text{Amb}}=(V^B_A,E^B_A,T^B_A,Hd^B_A),\quad & V^B_A=\big\{A_1\big\},\ A_1=\big\{\{r_2\},\{r_4,r_6\}\big\},\ E^B_A=\emptyset.\end{aligned}$$

Admissible set–relation on $\mathfrak{S}_2$***.*** *Define* R_{resp} *by*

$$(S,T)\in R_{\text{resp}}\iff \exists x\in\Omega\ \text{s.t.}\ \forall H\in S\cup T\ \exists U_H\in V(H)\ \exists g_H\in U_H\ \text{with}\ x\in g_H.$$

Intuition: the same responder x appears in at least one roster within every chosen agency.
Level $t=0$***: City-level MetaSuperHyperGraphs over*** $(\mathfrak{S}_2,\{R_{\text{resp}}\})$***.***

$$\mathsf{M}_A=(V_A,E_A,T_A,Hd_A,\lambda_A),\quad V_A=\{H^A_{\text{Fire}},H^A_{\text{Med}}\},$$

with one meta-hyperedge e_A*:*

$$T_A(e_A)=\{H^A_{\text{Fire}}\},\quad Hd_A(e_A)=\{H^A_{\text{Med}}\},\quad \lambda_A(e_A)=R_{\text{resp}}.$$

Witness: $x=r_2$ *(since* $r_2\in\{r_1,r_2\}\in U_1\in V^A_F$ *and* $r_2\in\{r_2,r_4\}\in M_1\in V^A_M$*).*
Similarly,

$$\mathsf{M}_B=(V_B,E_B,T_B,Hd_B,\lambda_B),\quad V_B=\{H^B_{\text{Police}},H^B_{\text{Amb}}\},$$

with one meta-hyperedge e_B*:*

$$T_B(e_B)=\{H^B_{\text{Police}}\},\quad Hd_B(e_B)=\{H^B_{\text{Amb}}\},\quad \lambda_B(e_B)=R_{\text{resp}},$$

witnessed *by* $x=r_2$ *(since* $r_2\in\{r_2,r_6\}\in P_1\in V^B_P$ *and* $r_2\in\{r_2\}\in A_1\in V^B_A$*).*

Level $t = 1$***: IMSHG (inter-city) over*** $(\mathfrak{S}_2^{(1)}, \mathscr{R}^{(1)}=\{R^{\Uparrow}_{\text{resp}}\})$***.*** *Let*

$$\mathsf{M}^{(1)} = (V^{(1)}, E^{(1)}, T^{(1)}, Hd^{(1)}, \lambda^{(1)}), \quad V^{(1)} = \{\mathsf{M}_A, \mathsf{M}_B\}.$$

Define one meta-hyperedge $E_{A\to B}$*:*

$$T^{(1)}(E_{A\to B}) = \{\mathsf{M}_A\}, \quad Hd^{(1)}(E_{A\to B}) = \{\mathsf{M}_B\}, \quad \lambda^{(1)}(E_{A\to B}) = R^{\Uparrow}_{\text{resp}}.$$

Incidence witness at depth 1. *Pick* $M = \mathsf{M}_A$*,* $N = \mathsf{M}_B$ *and*

$$A = \{H^A_{\text{Med}}\} \subseteq V_A, \qquad B = \{H^B_{\text{Amb}}\} \subseteq V_B.$$

Choose $x = r_2$*; then* $r_2 \in \{r_2, r_4\} \in M_1 \in V^A_M$ *and* $r_2 \in \{r_2\} \in A_1 \in V^B_A$*, so* $(A,B) \in R_{\text{resp}}$ *and*

$$\big(T^{(1)}(E_{A\to B}), Hd^{(1)}(E_{A\to B})\big) \in R^{\Uparrow}_{\text{resp}}.$$

Level $t = 2$***: IMSHG (regional) over*** $(\mathfrak{S}_2^{(2)}, \mathscr{R}^{(2)}=\{R^{\Uparrow\uparrow}_{\text{resp}}\})$***.*** *Create regional MSHGs whose vertices are city MSHGs:*

$$\mathsf{Reg}_{\text{North}}: \ V = \{\mathsf{M}_A\}, \ E = \emptyset; \qquad \mathsf{Reg}_{\text{South}}: \ V = \{\mathsf{M}_B\}, \ E = \emptyset.$$

*Define the depth-*2 *IMSHG*

$$\mathsf{M}^{(2)} = (V^{(2)}, E^{(2)}, T^{(2)}, Hd^{(2)}, \lambda^{(2)}), \quad V^{(2)} = \{\mathsf{Reg}_{\text{North}}, \mathsf{Reg}_{\text{South}}\},$$

with one meta-hyperedge $E_{\text{North}\to\text{South}}$*:*

$$T^{(2)}(E_{\text{North}\to\text{South}}) = \{\mathsf{Reg}_{\text{North}}\}, \quad Hd^{(2)}(E_{\text{North}\to\text{South}}) = \{\mathsf{Reg}_{\text{South}}\}, \quad \lambda^{(2)}(E_{\text{North}\to\text{South}}) = R^{\Uparrow\uparrow}_{\text{resp}}.$$

Incidence witness at depth 2. *Take* $M = \mathsf{M}_A \in V(\mathsf{Reg}_{\text{North}})$ *and* $N = \mathsf{M}_B \in V(\mathsf{Reg}_{\text{South}})$*. From the depth-*1 *verification,* $(M,N) \in R^{\Uparrow}_{\text{resp}}$ *via*

$$A = \{H^A_{\text{Med}}\}, \quad B = \{H^B_{\text{Amb}}\}, \quad x = r_2.$$

Hence

$$\big(T^{(2)}(E_{\text{North}\to\text{South}}), Hd^{(2)}(E_{\text{North}\to\text{South}})\big) \in R^{\Uparrow\uparrow}_{\text{resp}},$$

so $\mathsf{M}^{(2)}$ *is a valid depth-*2 *IMSHG linking regions by shared responder membership.*

Theorem 2.47 (IMSHG generalizes MetaSuperHyperGraph)**.** *Depth* 0 *IMSHGs are exactly MetaSuperHyperGraphs over* $(\mathfrak{S}_n, \mathscr{R})$*.*

Proof. By construction, $\mathfrak{S}_n^{(0)} = \mathfrak{S}_n$ and $\mathscr{R}^{(0)} = \mathscr{R}$. The definition of IMSHG with $t = 0$ is identical to the definition of a MetaSuperHyperGraph over $(\mathfrak{S}_n, \mathscr{R})$. □

Definition 2.48 (Unit 0-SuperHyperGraph and singleton embedding of relations)**.** *Let* $\mathsf{U}_0(x)$ *denote the* unit 0-SuperHyperGraph *on a base object x (single vertex x, no edges). For a binary relation R on a universe* $\mathfrak{U}$*, define the singleton embedding*

$$\iota_0(R) \ := \ \big\{ (\{x\}, \{y\}) \ \big| \ (x,y) \in R \big\},$$

viewed as a set–relation on $\mathscr{P}_{\text{fin}}(\mathfrak{U}) \times \mathscr{P}_{\text{fin}}(\mathfrak{U})$*.*

Lemma 2.49 (Lift commutes with singleton embedding on singletons)**.** *For any depth* $t \geq 0$*, any relation R on the depth-t universe, and any metalevel objects* M, N*,*

$$(\{M\}, \{N\}) \in \big(\iota_0(R)\big)^{\Uparrow} \quad \Longleftrightarrow \quad (M,N) \in R^{\uparrow}.$$

Proof. By the definition of $\Uparrow$ and ι_0,

$$(\{M\}, \{N\}) \in (\iota_0(R))^{\Uparrow} \iff \exists x \in V(M), y \in V(N): \ (\{x\}, \{y\}) \in \iota_0(R)$$

$$\iff \exists x \in V(M), y \in V(N): \ (x,y) \in R \iff (M,N) \in R^{\uparrow}.$$

□

Theorem 2.50 (IMSHG generalizes Iterated MetaHyperGraph)**.** *Fix any depth* $t \geq 0$*. Let an Iterated MetaHyperGraph (IMHG) of depth t over* $(\mathfrak{U}^{(t)}, \mathscr{Q}^{(t)})$ *be given (as defined earlier). Consider IMSHGs of depth t with* $n = 0$ *over* $\big(\mathfrak{S}_0^{(t)}, \iota_0(\mathscr{Q}^{(t)})\big)$*, and restrict to the full subcategory* $\mathbf{IMSHG}_1^{(t)}$ *in which every hyperedge e satisfies* $|T^{(t)}(e)| = |Hd^{(t)}(e)| = 1$ *and every vertex is of the form* $\mathsf{U}_0(\cdot)$*. Then* $\mathbf{IMSHG_1^{(t)}}$ *is canonically equivalent to the category of depth-t IMHGs over* $(\mathfrak{U}^{(t)}, \mathscr{Q}^{(t)})$*.*

Proof. Define mutually inverse functors F_t and G_t.
(F_t: IMHG → IMSHG) Let

$$\mathsf{H}^{(t)} = (V, E, S, H, \mu) \quad \text{(depth-}t\text{ IMHG over } (\mathfrak{U}^{(t)}, \mathscr{Q}^{(t)})).$$

Set

$$V' := \{\mathsf{U}_0(v) : v \in V\}, \quad E' := E,$$
$$T'(e) := \{\mathsf{U}_0(S(e))\}, \qquad Hd'(e) := \{\mathsf{U}_0(H(e))\}, \qquad \lambda'(e) := \iota_0\big(\mu(e)\big).$$

Incidence is preserved since

$$\big(T'(e), Hd'(e)\big) \in \iota_0(\mu(e)) \iff \big(\{S(e)\}, \{H(e)\}\big) \in \iota_0(\mu(e)) \iff (S(e), H(e)) \in \mu(e).$$

Thus $F_t(\mathsf{H}^{(t)}) = \mathsf{M}^{(t)} := (V', E', T', Hd', \lambda')$ is a depth-t IMSHG in $\mathbf{IMSHG}_1^{(t)}$.
(G_t: IMSHG → IMHG) Conversely, take

$$\mathsf{M}^{(t)} = (V', E', T', Hd', \lambda') \in \mathbf{IMSHG}_1^{(t)} \quad \text{over } (\mathfrak{S}_0^{(t)}, \iota_0(\mathscr{Q}^{(t)})).$$

Write $V := \{v \in \mathfrak{U}^{(t)} : \mathsf{U}_0(v) \in V'\}$, $E := E'$, and for each $e \in E$ let $S(e), H(e)$ be the unique elements with

$$T'(e) = \{\mathsf{U}_0(S(e))\}, \qquad Hd'(e) = \{\mathsf{U}_0(H(e))\}.$$

Since $\lambda'(e) \in \iota_0(\mathscr{Q}^{(t)})$, choose the unique $\mu(e) \in \mathscr{Q}^{(t)}$ with $\lambda'(e) = \iota_0(\mu(e))$. Incidence is equivalent by the same calculation:

$$(S(e), H(e)) \in \mu(e) \iff \big(\{S(e)\}, \{H(e)\}\big) \in \iota_0(\mu(e)) \iff \big(T'(e), Hd'(e)\big) \in \lambda'(e).$$

Hence $G_t(\mathsf{M}^{(t)}) = \mathsf{H}^{(t)} := (V, E, S, H, \mu)$ is a depth-t IMHG.
Finally, $G_t \circ F_t = \mathrm{id}$ and $F_t \circ G_t = \mathrm{id}$ on objects and morphisms by construction. Label compatibility across depths follows from the lemma: for any $R \in \mathscr{Q}^{(k)}$ and metalevel objects M, N at depth k,

$$(\{M\}, \{N\}) \in \big(\iota_0(R)\big)^{\Uparrow} \overset{\text{Lemma}}{\iff} (M, N) \in R^{\uparrow},$$

so lifted labels match at every level. Thus the categories are equivalent. □

3. Conclusion

In this paper, we formally defined the hypergraph analogue (MetaHyperGraph) and the superhypergraph analogue (MetaSuperHyperGraph) of MetaGraphs, and provided a concise discussion of their characteristics and illustrative applications. We also introduced iterative constructions such as the Iterated MetaGraph, representing a "graph of graphs of . . . of graphs," and briefly examined their properties and potential uses. It is hoped that future work will explore extensions incorporating frameworks such as Fuzzy Sets [49–52], Intuitionistic Fuzzy Sets[53–55], HyperFuzzy Sets[56–58], Hesitant Fuzzy Sets[59–62], picture fuzzy sets[63–65], Vague Sets[66–68], Neutrosophic Sets [33, 69–71], and Plithogenic Sets [72–74].

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Use of Generative AI and AI-Assisted Tools: We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

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