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Research Article

Constructing optimal fourth and eighth order iterative methods by using variant of Newton's method

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Abstract

In this paper, we have presented an optimal fourth order iterative method and an optimal eighth order iterative method without memory using weight functions. In terms of computational point of view, our first method require three evaluations (two function and one first derivatives) per iteration to get fourth order and the second method require four evaluations (three functions and one derivatives) per iteration to get eighth order. Hence, these methods have high efficiency indices 1.587 and 1.682 respectively. Some numerical examples are tested to know the performance of the new methods which verifies the theoretical results.

1. Introduction

It is known that a wide class of problems which arise in boundary value problems in Kinetic theory of gases, elasticity and other applied areas are mostly reduced to single variable nonlinear equations. One of the best root-finding methods for solving nonlinear scalar equation $f(x) = 0$ is Newton's iteration method. The local order of convergence of Newton's method is two and it is optimal with two function evaluations per iterative step. In recent years, numerous higher order iterative methods have been developed and analyzed for solving nonlinear equations that improve classical methods such as Newton's, Chebyshev, Chebyshev-Halley's, etc. As the order of convergence increases, so does the number of function evaluations per step. Hence, a new index to determine the efficiency called "Efficiency Index" (EI) is introduced in [9] to measure the balance between these quantities. Kung-Traub [5] conjectured that the order of convergence of any multi-point without memory method with d function evaluations cannot exceed the bound 2^{d-1}, the optimal order. Thus the optimal order for three evaluations per iteration would be four, four evaluations per iteration would be eight and so on.

Recently, some fourth and eighth order optimal I.F.s have been developed using weight functions (see [1, 2, 3, 6, 7, 8, 10, 12, 13, 15, 16, 19] and references therein). In [4, 14, 17, 18], third order method has been presented using the idea of mean and Newton iterations. In this paper, we have improved the order of the method to four (optimal) by using variants of third order Newton's method. Further, we have developed a family of eighth (optimal) order method using weight functions. In Section 2, some definitions are included which are required for our study and present the development of the new methods. Section 3 discusses the convergence analysis using MATHEMATICA. Section 4 presents few numerical examples and compare the results of the present methods with Newton's method and few optimal methods. Finally, section 5 gives conclusions on the present work.

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2. Development of the methods

If the sequence $\{x_n\}$ tends to a limit x^* in such a way that

$$\lim_{n\to\infty} \frac{x_{n+1}-x^*}{(x_n-x^*)^p} = C$$

for $p \geq 1$, then the order of convergence of the sequence is said to be p, and C is known as the asymptotic error constant. If $p=1$, $p=2$ or $p=3$, the convergence is said to be linear, quadratic or cubic, respectively.
Let $e_n = x_n - x^*$, then the relation

$$e_{n+1} = C\, e_n^p + O\left(e_n^{p+1}\right) = O\left(e_n^p\right). \tag{1}$$

is called the error equation. The value of p is called the order of convergence of the method. [9] The Efficiency Index is given by

$$EI = p^{\frac{1}{d}}, \tag{2}$$

where d is the total number of new function evaluations (the values of f and its derivatives) per iteration. Let $x_{n+1} = \psi(x_n)$ define an Iterative Function (I.F.). Let x_{n+1} be determined by new information at $x_n, \phi_1(x_n), \ldots, \phi_i(x_n), i \geq 1$. No old information is reused. Thus,

$$x_{n+1} = \psi(x_n, \phi_1(x_n), \ldots, \phi_i(x_n)). \tag{3}$$

Then ψ is called a multipoint I.F. without memory.
Kung-Traub Conjecture [5]
Let ψ be an I.F. without memory with d evaluations. Then

$$p(\psi) \leq p_{Opt} = 2^{d-1}, \tag{4}$$

where p_{opt} is the maximum order.
The Newton (also called Newton-Raphson) I.F. ($2^{nd}NR$) is given by

$$\psi_{2^{nd}NR}(x) = x - u(x),\, u(x) = \frac{f(x)}{f'(x)}. \tag{5}$$

The $2^{nd}NR$ I.F. is one-point I.F. with two function evaluations and it satisfies the Kung-Traub conjecture with $d=2$. Further, $EI_{2^{nd}NR} = 1.414$.
A family of third-order I.F. based on power means ($3^{rd}PM$) considered by Xiaojian [18] is given by

$$\psi_{3^{rd}PM}(x) = x - \frac{f(x)}{D(x,\beta)}, \quad D(x,\beta) = sign(f'(x))\left(\frac{f'(x)^\beta + (f'(x-u(x)))^\beta}{2}\right)^{\frac{1}{\beta}}. \tag{6}$$

The cases $\beta = 1, -1, 2$ correspond to arithmetic mean ($3^{rd}AM$), harmonic mean ($3^{rd}HM$) and square mean ($3^{rd}SM$) respectively. For the case $\beta = 0$, we consider $\beta \longrightarrow 0$ which is the geometric mean ($3^{rd}GM$) with
$D(x,0) = sign(f'(x))\sqrt{f'(x)f'(x-u(x))}$ since $\lim_{\beta\longrightarrow 0}\left(\frac{f'(x)^\beta+(f'(x-u(x)))^\beta}{2}\right)^{\frac{1}{\beta}} = \sqrt{f'(x)f'(x-u(x))}$. Let us consider the following third order method for the value of $\beta = 1$ in (6) (see [17]):

$$\psi_{3^{rd}AM}(x) = x - \frac{2f(x)}{f'(x) + f'(\psi_{2^{nd}NR}(x))}. \tag{7}$$

For $\beta = -1$ in (6) (see [4]):

$$\psi_{3^{rd}HM}(x) = x - \frac{f(x)}{2}\left(\frac{1}{f'(x)} + \frac{1}{f'(\psi_{2^{nd}NR}(x))}\right). \tag{8}$$

Also, consider Newton-Steffensen (NS) method with cubic convergence [14]

$$\psi_{3^{rd}NS}(x) = x - \frac{f(x)^2}{f'(x)[f(x) - f(\psi_{2^{nd}NR}(x))]}. \tag{9}$$

The family of methods (6) and (9) are of order three with three evaluations per full iteration having $EI = 1.442$. From literature survey, we observe that these methods (7), (8) and (9) are approximate equal interns of convergence order and EI. By equating these methods we can improve optimal fourth order method without weight functions as follows.
By equating (7) and (8), we have

$$f'(\psi_{2^{nd}NR}(x)) \approx \frac{f'(x)^2}{2f'(x) - f'(\psi_{2^{nd}NR}(x))}. \tag{10}$$

Also, by equating (8) and (9), we have

$$f'(\psi_{2^{nd}NR}(x)) \approx \frac{f'(x)[f(x) - f(\psi_{2^{nd}NR}(x))]}{f(x) + f(\psi_{2^{nd}NR}(x))} \tag{11}$$

Now, equating (10) and (11), we have

$$f'(\psi_{2^{nd}NR}(x)) \approx \frac{f'(x)[f(x)-3f(\psi_{2^{nd}NR}(x))]}{f(x)-f(\psi_{2^{nd}NR}(x))} \tag{12}$$

Equation (12) Substitute in (7), then we obtain new fourth-order I.F. as follows:

$$\psi_{4^{th}KM}(x) = x - \frac{f(x)}{f'(x)}\Big[\frac{f(x)-f(\psi_{2^{nd}NR}(x))}{f(x)-2f(\psi_{2^{nd}NR}(x))}\Big] \tag{13}$$

The efficiency of the proposed method (13) has improved where $EI = 1.587$ and this method known as Ostrowski's method [9]. Further, a family of optimal eighth order method from $4^{th}KM$ is proposed with 4 function evaluations using weight function.
Let us consider new family of optimal eighth order method as follows:

$$\psi_{8^{th}KM}(x) = \psi_{4^{th}KM}(x) - \frac{f(\psi_{4^{th}KM}(x))(\psi_{4^{th}KM}(x)-\psi_{2^{nd}NR}(x))}{f(\psi_{4^{th}KM}(x))-f(\psi_{2^{nd}NR}(x))}(G(\eta)\times H(\tau)), \tag{14}$$

where $G(\eta)$ and $H(\tau)$ are weight functions and expanding about 0, that should be chosen in order to obtain the eighth-order of convergence and $\eta = \frac{f(\psi_{4^{th}KM}(x))}{f(x)}$ and $\tau = \frac{f(\psi_{2^{nd}NR}(x))}{f(x)}$.
Next section, we analyze the convergence proof of the proposed family (14) with help of MATHEMATICA software.

3. Convergence Analysis

Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function having a simple root x^* in the open interval D, then the proposed family of I.F. $8^{th}KM$ (14) is eighth order convergence when $G(0) = 1$, $G'(0) = 2$, $H(0) = 1$, $H'(0) = 0$, $H''(0) = 2$ and $H'''(0) = 12$, being in this case the error equation

$$\psi_{8^{th}KM}(x) = \alpha + c_2^2(c_2^2-c_3)(7c_2^3-4c_2c_3+c_4)e_n^8 + O(e_n^9)$$

where $c_q = \frac{f^{(q)}(x^*)}{q!f'(x^*)}$, $q \geq 2$.

Proof. Let $e_n = x-\alpha$.
Using the Taylor series and we have

$$f(x) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + \ldots] \tag{15}$$

and

$$f'(x) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + \ldots] \tag{16}$$

where $c_q = \frac{f^{(q)}(x^*)}{q!f'(x^*)}$, $q \geq 2$. Now

$$\begin{aligned}\psi_{2^{nd}NR}(x) = {} & \alpha + c_2e_n^2 - 2(c_2^2-c_3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + (-8c_2^4 + 20c_2^2c_3 - 6c_3^2 - 10c_2c_4 + 4c_5)e_n^5 \\ & + (16c_2^5 - 52c_2^3c_3 + 33c_2c_3^2 + 28c_2^2c_4 - 17c_3c_4 - 13c_2c_5 + 5c_6)e_n^6 + \ldots\end{aligned} \tag{17}$$

Expanding $f(\psi_{2^{nd}NR}(x))$ about α and taking into account (17), we have

$$\begin{aligned}f(\psi_{2^{nd}NR}(x)) = {} & f'(\alpha)[c_2e_n^2 - 2(c_2^2-c_3)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 - 2(6c_2^4 - 12c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)e_n^5 \\ & + (28c_2^5 - 73c_2^3c_3 + 34c_2^2c_4 - 17c_3c_4 + c_2(37c_3^2 - 13c_5) + 5c_6)e_n^6 + \ldots]\end{aligned} \tag{18}$$

Using eqs. (15), (16) and (18) into (13), we obtain

$$\begin{aligned}\psi_{4^{th}KM}(x) = {} & \alpha + (c_2^3 - c_2c_3)e_n^4 - 2(2c_2^4 - 4c_2^2c_3 + c_3^2 + c_2c_4)e_n^5 + (10c_2^5 - 30c_2^3c_3 + 18c_2c_3^2 \\ & + 12c_2^2c_4 - 7c_3c_4 - 3c_2c_5)e_n^6 - 2(10c_2^6 - 40c_2^4c_3 - 6c_3^3 + 20c_2^3c_4 + 3c_4^2 + 8c_2^2(5c_3^2 - c_5) \\ & + 5c_3c_5 + c_2(-26c_3c_4 + 2c_6))e_n^7 + (36c_2^7 - 178c_2^5c_3 + 101c_2^4c_4 + 50c_3^2c_4 + 3c_2^3(84c_3^2 - 17c_5) \\ & - 17c_4c_5 13c_3c_6 + c_2^2(-209c_3c_4 + 20c_6) + c_2(-91c_3^3 + 37c_4^2 + 68c_3c_5 - 5c_7))e_n^8 + \ldots\end{aligned} \tag{19}$$

Expanding $f(\psi_{4^{th}KM}(x))$ about α and taking into account (19), we have

$$\begin{aligned}f(\psi_{4^{th}KM}(x)) = {} & f'(\alpha)\Big[(c_2^3 - c_2c_3)e_n^4 - 2(2c_2^4 - 4c_2^2c_3 + c_3^2 + c_2c_4)e_n^5 + (10c_2^5 - 30c_2^3c_3 + 18c_2c_3^2 \\ & + 12c_2^2c_4 - 7c_3c_4 - 3c_2c_5)e_n^6 - 2(10c_2^6 - 40c_2^4c_3 - 6c_3^3 + 20c_2^3c_4 + 3c_4^2 + 8c_2^2(5c_3^2 - c_5) \\ & + 5c_3c_5 + c_2(-26c_3c_4 + 2c_6))e_n^7 + (37c_2^7 - 180c_2^5c_3 + 101c_2^4c_4 + 50c_3^2c_4 + c_2^3(253c_3^2 - 51c_5) \\ & - 17c_4c_5 - 13c_3c_6 + c_2^2(-209c_3c_4 + 20c_6) + c_2(-91c_3^3 + 37c_4^2 + 68c_3c_5 - 5c_7))e_n^8 + \ldots\Big]\end{aligned} \tag{20}$$

Using equations (17)-(20) into (14), we obtained

$$\begin{aligned}
\psi_{8^{th}KM}(x) &= \alpha - c_2(c_2^2-c_3)\Big(-1+G(0)H(0)\Big)e_n^4 + \Big(2c_3^2(-1+G(0)H(0)) + 2c_2c_4(-1+G(0)H(0)) \\
&+ c_2^4(-4+4G(0)H(0)-G(0)H'(0)) + c_2^2c_3(8+G(0)(-8H(0)+H'(0)))\Big)e_n^5 \\
&+ \Big(7c_3c_4(-1+G(0)H(0)) + 2c_2^2c_4(6+G(0)(-6H(0)+H'(0))) + c_2(3c_5(-1+G(0)H(0)) \\
&+ c_3^2(18-18G(0)H(0)+4G(0)H'(0))) + c_2^5(10+G(0)(-9H(0)+7H'(0)-\frac{H''(0)}{2})) \\
&+ \frac{1}{2}c_2^3c_3(-60+G(0)(58H(0)-26H'(0)+H''(0)))\Big)e_n^6 + \Big(2(3c_4^2(-1+G(0)H(0)) \\
&+ 5c_3c_5(-1+G(0)H(0)) + c_3^3(6-6G(0)H(0)+2G(0)H'(0))) + c_2(4c_6(-1+G(0)H(0)) \\
&+ 2c_3c_4(26-26G(0)H(0)+7G(0)H'(0))) + c_2^3c_4(-40+G(0)(38H(0)-21H'(0)+H''(0))) \\
&+ c_2^2(c_5(16-16G(0)H(0)+3G(0)H'(0)) + c_3^2(-80+76G(0)H(0)-G'(0)H(0)-50G(0)H'(0) \\
&+ 3G(0)H''(0))) + c_2^6(-20-G'(0)H(0)+G(0)(14H(0)-29H'(0)+5H''(0)-\frac{H'''(0)}{6})) \\
&+ \frac{1}{6}c_2^4c_3(12(40+G'(0)H(0)) + G(0)(-408H(0)+474H'(0)-54H''(0)+H'''(0)))\Big)e_n^7 \\
&+ \frac{1}{6}\Big(102c_4c_5(-1+G(0)H(0)) + 78c_3c_6(-1+G(0)H(0)) + 60c_3^2c_4(5-5G(0)H(0) \\
&+ 2G(0)H'(0)) + 6c_2(5c_7(-1+G(0)H(0)) + 4c_3c_5(17-17G(0)H(0)+5G(0)H'(0)) \\
&+ c_4^2(37-37G(0)H(0)+12G(0)H'(0)) + c_3^3(-91-4G'(0)H(0)+G(0)(87H(0) \\
&- 76H'(0)+6H''(0)))) + 3c_2^2(8c_6(5+G(0)(-5H(0)+H'(0))) + c_3c_4(-418-8G'(0)H(0) \\
&+ G(0)(390H(0)-310H'(0)+21H''(0)))) + c_2^5c_3(-12(89+13G'(0)H(0)-G'(0)H'(0)) \\
&+ G(0)(690H(0)-2004H'(0)+450H''(0)-23H'''(0))) + 2c_2^4c_4(303+12G'(0)H(0) \\
&+ G(0)(-246H(0)+348H'(0)-45H''(0)+H'''(0))) + 6c_2^7(36+9G'(0)H(0)-G'(0)H'(0) \\
&+ G(0)(-15H(0)+93H'(0)-29H''(0)+(13\frac{H'''(0)}{6}))) + c_2^3(3c_5(-102+G(0)(96H(0) \\
&- 58H'(0)+3H''(0))) + 2c_3^2(756+63G'(0)H(0)-3G'(0)H'(0) \\
&+ G(0)(-624H(0)+987H'(0)-147H''(0)+4H'''(0))))\Big)e_n^8 + \ldots
\end{aligned}$$

By choice of $G(0)=1$, $G'(0)=2$, $H(0)=1$, $H'(0)=0$, $H''(0)=2$ and $H'''(0)=12$ in above equation. We have, finally

$$\psi_{8^{th}KM}(x) = \alpha + c_2^2(c_2^2-c_3)(7c_2^3-4c_2c_3+c_4)e_n^8 + O(e_n^9) \tag{21}$$

which proves the proposed family (14) convergent with order eight. □

By some conditions on the weight function have been imposed to assure the desired order of convergence. This gives us the possibility of designing different schemes depending on the $|H^{(k)}(0)| < \infty, k = 4,5...$ used, as for example

$$G(\eta) = 1+2\eta, \quad H(\tau) = 1+\tau^2+2\tau^3.$$

or

$$G(\eta) = 1+2\eta, \quad H(\tau) = 1+\tau^2+2\tau^3+\tau^4.$$

or

$$G(\eta) = 1+2\eta, \quad H(\tau) = 1+\tau^2+2\tau^3+\tau^4+\tau^5.$$

We use the first one in the rest of the manuscript, denoting the resulting scheme by $8^{th}KM$. Thus, proposed family of I.F. (14) is optimal eighth order and its efficiency has improved $EI = 1.682$.

4. Numerical examples

In this section, numerical results on some test functions are compared for the new methods $4^{th}KM$ and $8^{th}KM$ with some existing eighth order methods and $2^{nd}NR$. Numerical computations have been carried out in the MATLAB software with 500 significant digits. Depending on the precision of the computer, we have used the stopping criteria for the iterative process either $|f(x_N)| < \varepsilon$ or $error = |x_N - x_{N-1}| < \varepsilon$ where $\varepsilon = 10^{-50}$ and N is the number of iterations required for convergence. The computational order of convergence is given by

$$\rho = \frac{\ln|(x_N - x_{N-1})/(x_{N-1}-x_{N-2})|}{\ln|(x_{N-1}-x_{N-2})/(x_{N-2}-x_{N-3})|}.$$

We consider the following iterative methods for solving scalar nonlinear equations for the purpose of comparison: Liu-Wang Method (8^{th}LWM) [6]:

$$y = x - \frac{f(x)}{f'(x)}, z = y - \frac{f(x)}{f(x)-2f(y)}\frac{f(y)}{f'(x)},$$
$$\psi_{8^{th}LWM}(x) = z - \frac{f(z)}{f'(x)}\left(\left(\frac{f(x)-f(y)}{f(x)-2f(y)}\right)^2 + \frac{f(z)}{f(y)-f(z)} + \frac{4f(z)}{f(x)+f(z)}\right). \quad (22)$$

Petkovic-Neta-Petkovic-Dzunic Method (8^{th}PNPDM) [11]:

$$y = x - \frac{f(x)}{f'(x)}, z = x - \left(\left(\frac{f(y)}{f(x)}\right)^2 - \frac{f(x)}{f(y)-f(x)}\right)\frac{f(x)}{f'(x)},$$
$$\psi_{8^{th}PNPDM}(x) = z - \frac{f(z)}{f'(x)}\left(\varphi(t) + \frac{f(z)}{f(y)-f(z)} + \frac{4f(z)}{f(x)}\right), \quad (23)$$
$$\text{where } \varphi(t) = 1+2t+2t^2-t^3 \text{ and } t = \frac{f(y)}{f(x)}.$$

Sharma-Arora Method (8^{th}SAM) [13]:

$$y = x - \frac{f(x)}{f'(x)}, z = y - \left(3 - 2\frac{f[y,x]}{f'(x)}\right)\frac{f(y)}{f'(x)},$$
$$\psi_{8^{th}SAM}(x) = z - \frac{f(z)}{f'(x)}\left(\frac{f'(x)-f[y,x]+f[z,y]}{2f[z,y]-f[z,x]}\right). \quad (24)$$

The following test functions and their simple zeros for our study are given below:

$f_1(x) = \sin(2\cos x) - 1 - x^2 + e^{\sin(x^3)}$, $x^* = -0.7848959876612125352...$

$f_2(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$, $x^* = -1.2076478271309189270...$

$f_3(x) = x^3 + 4x^2 - 10$, $x^* = 1.3652300134140968457...$

$f_4(x) = sin(x) + cos(x) + x$, $x^* = -0.4566247045676308244...$

$f_5(x) = \frac{x}{2} - \sin x$, $x^* = 1.8954942670339809471...$

$f_6(x) = (x+2)e^x - 1$, $x^* = -0.4428544010023885831...$

$f_7(x) = x^2 + \sin(\frac{x}{5}) - \frac{1}{4}$, $x^* = 0.4099920179891371316...$

Table 1 shows the corresponding results for $f_1(x)$ to $f_7(x)$. If the initial points are very close to the root, then we obtain least number of iterations and lowest error. Hence, the present new method $8^{th}KM$ has better efficiency as compared to $2^{nd}NR$, $4^{th}KM$, $8^{th}LWM$, $8^{th}PNPDM$, $8^{th}SAM$. Specifically when consider the function $f_3(x), x_0 = 0.5$, $8^{th}PNPDM$, $8^{th}SAM$ methods are converges badly.

5. Conclusion

In this work, we have proposed a family of optimal three-point eighth order methods using weight functions. It is clear that our proposed new family of methods require only four evaluations per iterative step to obtain eighth order method. To illustrate the proposed new methods and to check the validity of the theoretical results we have tabulated numerical results. The performance is compared with Newton's method and some recently developed methods and proposed method is to be superior over some existing methods.

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Table 1: Numerical results for test functions

$f(x)$	Methods	N	$error$	ρ	$cpu(s)$
$f_1(x), x_0 = -0.9$	$2^{nd}NR$	7	7.7e-074	1.99	0.643284
	$4^{th}KM$	4	6.0e-067	3.99	0.490021
	$8^{th}LWM$	3	4.5e-059	7.91	0.519256
	$8^{th}PNPDM$	3	8.8e-056	7.87	0.492658
	$8^{th}SAM$	3	3.4e-060	7.88	0.482706
	$8^{th}KM$	3	1.0e-062	7.91	0.482377
$f_2(x), x_0 = -1.7$	$2^{nd}NR$	9	4.3e-054	1.99	0.648932
	$4^{th}KM$	5	4.4e-093	4.00	0.602758
	$8^{th}LWM$	4	2.3e-125	8.00	0.624357
	$8^{th}PNPDM$	4	1.0e-057	7.97	0.615154
	$8^{th}SAM$	4	3.8e-093	8.00	0.604951
	$8^{th}KM$	4	8.3e-121	7.99	0.604853
$f_3(x), x_0 = 0.5$	$2^{nd}NR$	9	5.3e-058	1.99	0.543270
	$4^{th}KM$	5	1.1e-068	3.99	0.534235
	$8^{th}LWM$	5	1.3e-098	7.99	0.591683
	$8^{th}PNPDM$	19	7.1e-114	7.99	1.557381
	$8^{th}SAM$	342	5.3e-148	8.00	31.346577
	$8^{th}KM$	5	8.9e-203	7.99	0.504832
$f_4(x), x_0 = -0.2$	$2^{nd}NR$	7	6.8e-096	1.99	0.500082
	$4^{th}KM$	4	1.1e-076	3.99	0.415800
	$8^{th}LWM$	3	1.0e-068	8.08	0.464638
	$8^{th}PNPDM$	3	3.5e-067	8.10	0.455332
	$8^{th}SAM$	3	8.0e-069	8.06	0.421791
	$8^{th}KM$	3	2.1e-073	8.08	0.419979
$f_5(x), x_0 = 1.6$	$2^{nd}NR$	8	6.8e-087	1.99	0.595717
	$4^{th}KM$	5	2.5e-168	4.00	0.457531
	$8^{th}LWM$	4	1.5e-242	7.99	0.484437
	$8^{th}PNPDM$	4	3.1e-215	8.00	0.486276
	$8^{th}SAM$	4	3.7e-183	8.00	0.496179
	$8^{th}KM$	4	6.6e-250	7.99	0.474353
$f_6(x), x_0 = -0.3$	$2^{nd}NR$	7	7.7e-066	1.99	0.477958
	$4^{th}KM$	4	4.1e-074	3.99	0.427945
	$8^{th}LWM$	3	4.4e-064	7.99	0.447357
	$8^{th}PNPDM$	3	1.5e-051	7.94	0.433124
	$8^{th}SAM$	3	3.5e-056	7.93	0.412904
	$8^{th}KM$	3	4.0e-064	7.96	0.411605
$f_7(x), x_0 = 0.2$	$2^{nd}NR$	8	8.2e-076	2.00	0.601464
	$4^{th}KM$	5	7.4e-151	3.99	0.476424
	$8^{th}LWM$	4	1.9e-202	7.99	0.475347
	$8^{th}PNPDM$	4	1.5e-074	8.02	0.487462
	$8^{th}SAM$	4	2.5e-128	8.00	0.479096
	$8^{th}KM$	4	4.5e-205	7.99	0.467054

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