Jagua Publication

Current Research in Interdisciplinary Studies 2(4) (2023): 1 – 15. **www.jpub.org**

Research Article

New higher order iterative methods for solving nonlinear equations and their basins of attraction

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Received: March 21, 2023
Accepted: April 15, 2023
Published: April 28, 2023

How to cite:
Madhu. 2023. " New higher order iterative methods for solving nonlinear equations and their basins of attraction." *Current Research in Interdisciplinary Studies* 2(4): 1 – 15. **https://doi.org/10.58614/cris241**

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Abstract

In this work, we have developed a family of eighth order Newton-type methods for solving nonlinear equations with four functional evaluations. Further, we have improved a new member of eighth order method to sixteenth order iterative method with five function evaluations. These two methods are satisfying Kung-Traub Conjecture and their efficiency indices are 1.682 and 1.741. Numerical results are carried out to validate the new methods and some existing compared methods. We consider a concrete variety of real life problems such as projectile motion, Planck's radiation law, the trajectory of electron in air gap two parallel plates between, Van der Waal's equation which explains the behavior of a real gas by introducing in the ideal gas equations, in order to check the applicability and effectiveness of our proposed methods. Finally, the basins of attraction are also given to demonstrate their dynamical behavior in the complex plane.

AMS Subject Classification: 65H05 · 65D05 · 41A25
Key Words: kung-traub conjecture, multi-point iterations, non-linear equation, optimal order, basins of attraction.

1 Introduction

One of the most important problem in engineering, scientific computing and applied mathematics, in general, is the problem of solving a nonlinear equation $f(x) = 0$. The most familiar method of solving non linear equation is Newton's iteration method. The local order of convergence of Newton's method is two and it is an optimal method with two function evaluation per iterative step. In the past decade, several higher order iterative methods have been developed and analyzed for solving nonlinear equations that improve classical methods such as Newton's method, Chebyshev method, Halley's iteration method, etc. As the order of convergence increases, so does the number of function evaluations per step. Hence, a new index to determine the efficiency called efficiency index is introduced in (Ostrowski 1960) to measure the balance between these quantities.

The main motive of this work is to implement a efficient algorithms for finding the solution of nonlinear equations. We obtained an optimal iterative methods which will supports the conjecture (Kung and Traub 1974). Kung-Traub Conjectured state that the multi-point iteration scheme without memory based on d functional evaluations, could achieve an optimal convergence order 2^{d-1}. Further, we studied the behaviour of iterative scheme in the complex plane. Also, many authors have worked with these idea on different iterative scheme (Amat, Busquier, and Plaza 2004a, 2004b; Babajee and Madhu 2019; Cordero *et al.* 2010; Curry, Garnett, and Sullivan 1983; Soleymani, Babajee, and Sharifi 2012; Tao and Madhu 2019; Vrscay 1986; Vrscay and Gilbert 1987) described the basin of attraction of some well-known iterative scheme.

Rest of the paper is organized as follows. In Section 2 the new family is developed and its convergence analysis is discussed. In section 3, we display the performance of proposed methods and other compared methods described by numerical examples. In section 4, solve some real world applications of different nature to show the effectiveness of the proposed schemes. In section 5, the proposed methods are studied in the complex plane using basins of attraction. Section 6 gives concluding remarks.

2 Development of methods and its Convergence

In our study, through weight function procedure with Newton's scheme, a new family of iterative method is presented, with the following iterative expression:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, w_n = y_n - \frac{1}{H(\tau)}\frac{f(y_n)}{f'(x_n)}, \tau = \frac{f(y_n)}{f(x_n)}. \tag{1}$$

Next, we prove the convergence analysis of proposed methods with help of MATHEMATICA software. Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function having continuous derivatives. If $f(x)$ has a simple root x^* in the open interval D and x_0 is chosen in a sufficiently small neighborhood of x^*, then the family of method (1) is of fourth order convergence, when

$$H(0) = 1,\ H'(0) = -2,\ |H''(0)| < \infty \tag{2}$$

and it satisfies the error equation:

$$e_{n+1} = d[0]e_n^4 + O(e_n^5),\ \ d[0] = (1 + H''(0))c[2]^3 - c[2]c[3]). \tag{3}$$

$c[j] = \dfrac{f^{(j)}(x^*)}{j!f'(x^*)}$, $j = 2, 3, 4, \ldots$ and $e_n = x_n - x^*$. Further, our main motive is to develop optimal eighth order methods with help of fourth order methods, a new family of eighth order iterative methods is presented, with the following iterative expression

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},\ \ w_n = y_n - \frac{1}{H(\tau)}\frac{f(y_n)}{f'(x_n)},\ \ x_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)}, \tag{4}$$

where the method having 8th order convergence with 5 function evaluations. Hence, this is non optimal method. To obtain an optimal scheme, need to minimize the number of function evaluations and keep the same convergence order 8, and so we estimate $f'(w_n)$ by the following polynomial

$$\chi(t) = a_0 + a_1(t - x) + a_2(t - x)^2 + a_3(t - x)^3, \tag{5}$$

which satisfies

$$\chi(x_n) = f(x_n),\ \chi'(x_n) = f'(x_n),\ \chi(y_n) = f(y_n),\ \chi(w_n) = f(w_n).$$

Let us define the divided differences

$$f[y_n, x_n] = \frac{f(y_n) - f(x_n)}{y_n - x_n},\ f[y_n, x_n, x_n] = \frac{f[y_n, x_n] - f'(x_n)}{y_n - x_n}.$$

On implementing the above conditions on (5), we obtain four linear equations with four unknowns a_0, a_1, a_2 and a_3. From $\chi(x_n) = f(x_n)$, $\chi'(x_n) = f'(x_n)$, we get $a_0 = f(x_n)$ and $a_1 = f'(x_n)$. To find a_2 and a_3, we solve the following equations:

$$\begin{aligned} f(y_n) &= f(x_n) + f'(x_n)(y_n - x_n) + a_2(y_n - x_n)^2 + a_3(y_n - x_n)^3 \\ f(w_n) &= f(x_n) + f'(x_n)(w_n - x_n) + a_2(w_n - x_n)^2 + a_3(w_n - x_n)^3. \end{aligned}$$

Solving above equations by using divided difference, we have

$$a_2 = \frac{f[y_n, x_n, x_n](w_n - x_n) - f[w_n, x_n, x_n](y_n - x_n)}{w_n - y_n}, \; a_3 = \frac{f[w_n, x_n, x_n] - f[y_n, x_n, x_n]}{w_n - y_n}. \tag{6}$$

Further, using eq. (6), we have the estimation

$$f'(w_n) \approx \chi'(w_n) = f'(x_n) + 2a_2(w_n - x_n) + 3a_3(w_n - x_n)^2.$$

Finally, we propose a new family of optimal eighth order method

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ w_n = y_n - \frac{1}{H(\tau)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = w_n - \frac{f(w_n)}{f'(x_n) + 2a_2(w_n - x_n) + 3a_3(w_n - x_n)^2}, \end{cases} \tag{7}$$

where a_2 and a_3 are given in (6).

Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function having continuous derivatives. If $f(x)$ has a simple root x^* in the open interval D and x_0 chosen in sufficiently small neighborhood of x^*, then the method (7) is of local eighth order convergence and and it satisfies the error equation

$$e_{n+1} = c[2]d[0]\Big(c[4] + d[0]\Big)e_n^8 + O(e_n^9)$$

Let $\tilde{e}_n = y_n - x^*$, $\hat{e}_n = w_n - x^*$ and $c[j] = \dfrac{f^{(j)}(x^*)}{j!f'(x^*)}$, $j = 2, 3, 4, \ldots$. Expanding $f(x_n)$ and $f'(x_n)$ about x^* by Taylor's method, we have

$$f(x_n) = f'(x^*)\Big(e_n + c[2]e_n^2 + c[3]e_n^3 + c[4]e_n^4 + c[5]e_n^5 + c[6]e_n^6 + c[7]e_n^7 + c[8]e_n^8 + O(e_n^9)) \tag{8}$$

and

$$f'(x_n) = f'(x^*)\Big(1 + 2c[2]e_n + 3c[3]e_n^2 + 4c[4]e_n^3 + 5c[5]e_n^4 + 6c[6]e_n^5 + 7c[7]e_n^6 + 8c[8]e_n^7 + 9c[9]e_n^8 + O(e_n^9)\Big) \tag{9}$$

Thus,

$$\begin{aligned} \tilde{e} &= c[2]e_n^2 + \Big(-2c[2]^2 + 2c[3]\Big)e_n^3 + \Big(4c[2]^3 - 7c[2]c[3] + 3c[4]\Big)e^4 + \Big(-8c[2]^4 \\ &+ 20c[2]^2c[3] - 6c[3]^2 - 10c[2]c[4] + 4c[5]\Big)e^5 + \Big(16c[2]^5 - 52c[2]^3c[3] + 28c[2]^2c[4] - 17c[3]c[4] \\ &+ c[2](33c[3]^2 - 13c[5]) + 5c[6]\Big)e^6 - 2\Big(16c[2]^6 - 64c[2]^4c[3] - 9c[3]^3 + 36c[2]^3c[4] + 6c[4]^2 + 9c[2]^2(7c[3]^2 \\ &- 2c[5]) + 11c[3]c[5] + c[2](-46c[3]c[4] + 8c[6]) - 3c[7]\Big)e^7 + \Big(64c[2]^7 - 304c[2]^5c[3] \\ &+ 176c[2]^4c[4] + 75c[3]^2c[4] + c[2]^3(408c[3]^2 - 92c[5]) - 31c[4]c[5] - 27c[3]c[6] \\ &+ c[2]^2(-348c[3]c[4] + 44c[6]) + c[2](-135c[3]^3 + 64c[4]^2 + 118c[3]c[5] - 19c[7]) + 7c[8]\Big)e^8 + \ldots. \end{aligned} \tag{10}$$

Expanding $f(y_n)$ about x^* by Taylor's method, we have

$$f(y_n) = f'(x^*)\Big(\tilde{e} + c[2]\tilde{e}_n^2 + c[3]\tilde{e}_n^3 + c[4]\tilde{e}_n^4 + O(\tilde{e}_n^5)\Big) \tag{11}$$

Also, expanding $f(w_n)$ about x^* by Taylor's method, we have

$$f(w_n) = f'(x^*)\Big(\hat{e} + c[2]\hat{e}_n^2 + O(\hat{e}_n^3)\Big) \tag{12}$$

Substituting equations (3)-(12) in the third step of (7) and simplifying, we obtain

$$e_{n+1} = c[2]d[0]\Big(c[4] + d[0]\Big)e_n^8 + O(e_n^9).$$

This reveals that the proposed family of methods attains eighth-order convergence. By choice of any value of $H''(0)$ in (2), we obtain a new optimal eighth order methods. Therefore, some simple members of the family (7) are as follows. Proposed method ($PM1$): By choosing $H''(0) = 0$, we obtain as

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ w_n = y_n - \left(\frac{1}{1-2\tau}\right)\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = w_n - \frac{f(w_n)}{f'(x_n)+2a_2(w_n-x_n)+3a_3(w_n-x_n)^2}, \end{cases} \tag{13}$$

This method has the following error equation $e_{n+1} = c[2]^2\Big(c[2]^2\Big(c[2]^2 - c[3]\Big)\Big(c[2]^3 - c[2]c[3] + c[4]\Big)\Big)e_n^8 + O(e_n^9)$. Proposed method ($PM2$): By choosing $H''(0) = -2$, we obtain as

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ w_n = y_n - \left(\frac{1}{1-2\tau-\tau^2}\right)\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = w_n - \frac{f(w_n)}{f'(x_n)+2a_2(w_n-x_n)+3a_3(w_n-x_n)^2}, \end{cases} \tag{14}$$

This method has the following error equation $e_{n+1} = c[2]^2c[3]\Big(c[2]c[3] - c[4]\Big)e_n^8 + O(e_n^9)$. Proposed method ($PM3$): By choosing $H''(0) = -4$, we obtain as

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ w_n = y_n - \left(\frac{1}{1-2\tau-2\tau^2}\right)\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = w_n - \frac{f(w_n)}{f'(x_n)+2a_2(w_n-x_n)+3a_3(w_n-x_n)^2}, \end{cases} \tag{15}$$

This method has the following error equation $e_{n+1} = c[2]^2\Big(c[2]^2 + c[3]\Big)\Big(c[2]^3 + c[2]c[3] - c[4]\Big)e_n^8 + O(e_n^9)$.

3 Numerical Examples

In this section, numerical results on some test functions are compared for the new methods $PM1$, $PM2$ and $PM3$ with some existing eighth order methods and Newton's method. Numerical computations have been carried out in the MATLAB software with 500 significant digits. We have used the stopping criteria for the iterative process satisfying $error = |x_N - x_{N-1}| < \epsilon$, where $\epsilon = 10^{-50}$ and N is the number of iterations required for convergence. The computational order of convergence is given by ((Cordero and Torregrosa 2007))

$$\rho = \frac{\ln|(x_N - x_{N-1})/(x_{N-1} - x_{N-2})|}{\ln|(x_{N-1} - x_{N-2})/(x_{N-2} - x_{N-3})|}.$$

We consider the following iterative methods for solving nonlinear equations for the purpose of comparison: LWM method proposed by Liu et al (Liu and Wang 2010),

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ w_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = w_n - \frac{f(w_n)}{f'(x_n)}\left(\left(\frac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)}\right)^2 + \frac{f(w_n)}{f(y_n)-f(w_n)} + \frac{4f(w_n)}{f(x_n)+f(w_n)}\right). \end{cases} \tag{16}$$

SAM method proposed by Sharma et al (Sharma and Arora 2014),

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ w_n = y_n - \left(3 - 2\frac{f[y_n,x_n]}{f'(x_n)}\right)\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = w_n - \frac{f(w_n)}{f'(x_n)}\left(\frac{f'(x_n)-f[y_n,x_n]+f[w_n,y_n]}{2f[w_n,y_n]-f[w_n,x_n]}\right). \end{cases} \tag{17}$$

$CFGT$ method proposed by Cordero et al (**cfgt-2014**),

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ w_n = y_n - \frac{f(y_n)}{f'(x_n)}\frac{1}{1-2t+t^2-t^3/2}, \\ x_{n+1} = w_n - \frac{1+3r}{1+r}\frac{f(w_n)}{f[w_n,y_n]+f[w_n,x_n,x_n](w_n-y_n)}, t = \frac{f(y_n)}{f(x_n)}, r = \frac{f(w_n)}{f(x_n)}. \end{cases} \tag{18}$$

CTV method proposed by Cordero et al (Cordero, Torregrosa, and Vasileva 2011),

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ w_n = x_n - \frac{1-t}{1-2t}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = w_n - \left(\frac{1-t}{1-2t} - v\right)^2 \frac{1}{1-3v}\frac{f(w_n)}{f'(x_n)}, \ v = \frac{f(w_n)}{f(y_n)}. \end{cases} \tag{19}$$

NCS method proposed by Neta et al (Neta, Chun, and Scott 2014),

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ w_n = y_n - \frac{1}{(1-t)^2}\frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = w_n - \frac{f(w_n)}{H_3'(w_n)}, \\ H_3'(w_n) = 2(f[x_n,w_n] - f[x_n,y_n]) + f[y_n,w_n] + \frac{y_n-w_n}{y_n-x_n}(f[x_n,y_n] - f'(x_n)) \end{cases} \tag{20}$$

The following test functions and their simple zeros for our study are given below:

$$\begin{cases} f_1(x) = \sin(2\cos x) - 1 - x^2 + e^{\sin(x^3)}, & x^* = -0.7848959876612125352... \\ f_2(x) = x^3 + 4x^2 - 10, & x^* = 1.3652300134140968457... \\ f_3(x) = sin(x) + cos(x) + x, & x^* = -0.4566247045676308244... \\ f_4(x) = x^2 + sin(\frac{x}{5}) - \frac{1}{4}, & x^* = 0.4099920179891371316... \end{cases}$$

In table 1 show the results for all the test functions for a given initial point. The computational order of convergence conforms with theoretical convergence order. When the starting points are close to the zero, then we obtain less number of iterations with least error. When the starting points are far away to the zero, then we may not get least error. We observe that the new methods in all the test function have better efficiency as compared to other existing methods of the same order.

4 Applications to some real world problems

Example 1: We consider the classical projectile problem (Babajee and Madhu 2019; Kantrowitz and Neumann 2014) in which a projectile is launched from a tower of height $h > 0$, with initial speed v and at an angle θ with respect to the horizontal onto a hill, which is defined by the function ω, called the impact function which is dependent on the horizontal distance, x. We wish to find the optimal launch angle θ_m which maximizes the horizontal distance. In our calculations, we neglect air resistances.

I.F.	N	$\|x_1 - x_0\|$	$\|x_2 - x_1\|$	$\|x_3 - x_2\|$	$\|x_N - x_{N-1}\|$	ρ	TNFE	CPU_{time}
$f_1, x_0 = -1.0$								
NM	7	0.1967	0.0182	2.3636e-04	9.0696e-61	2.00	14	1.144847
LWM	4	0.2151	2.1060e-06	1.7919e-45	0	7.82	16	0.979315
SAM	4	0.2151	1.3780e-06	6.7387e-47	0	7.80	16	0.989816
$CFGT$	3	0.2151	1.8564e-07	3.3933e-55	3.3933e-55	7.87	12	0.811550
CTV	4	0.2151	2.1081e-06	7.2218e-46	0	7.90	16	1.015022
NCS	3	0.2151	2.6854e-07	1.4762e-53	1.4762e-53	7.83	12	0.777970
$PM1$	3	0.2151	1.6751e-07	8.8559e-56	8.8559e-56	7.90	12	0.777329
$PM2$	3	0.2151	8.7696e-08	1.1049e-58	1.1049e-58	7.96	12	0.730283
$PM3$	3	0.2151	3.0044e-08	1.4996e-62	1.4996e-62	7.92	12	0.743219
$f_2, x_0 = 0.9$								
NM	8	0.6263	0.1497	0.0113	1.3514e-72	2.00	16	1.023982
LWM	4	0.4660	7.3967e-04	3.9688e-27	2.7346e-213	7.99	16	0.740913
SAM	4	0.4492	0.0160	7.4896e-16	1.5396e-122	7.99	16	0.731669
$CFGT$	4	0.4654	1.8025e-04	6.6893e-33	2.4091e-260	7.99	16	0.810392
CTV	4	0.4652	6.6285e-05	2.2445e-36	3.8782e-288	7.99	16	0.746914
NCS	4	0.4655	2.8120e-04	8.1224e-31	3.9399e-243	7.99	16	0.764206
$PM1$	4	0.4653	3.3164e-05	5.5734e-39	3.5460e-309	7.99	16	0.710881
$PM2$	4	0.4653	6.9411e-05	2.3267e-37	3.7105e-297	7.99	16	0.724889
$PM3$	4	0.4660	7.2943e-04	8.5164e-28	2.9522e-219	7.99	16	0.723621
$f_3, x_0 = -0.9$								
NM	6	0.4415	0.0019	3.5152e-07	1.9715e-59	2.00	12	0.912788
LWM	3	0.4434	5.7199e-10	1.4223e-78	1.4223e-78	7.71	12	0.793040
SAM	3	0.4434	1.4115e-08	2.4782e-67	2.4782e-67	7.83	12	0.784606
$CFGT$	3	0.4434	2.0940e-10	2.5879e-83	2.5879e-83	7.81	12	0.819606
CTV	3	0.4434	4.2747e-08	3.6452e-63	3.6452e-63	7.84	12	0.830108
NCS	3	0.4434	8.0225e-11	5.6893e-87	5.6893e-87	7.81	12	0.791882
$PM1$	3	0.4434	8.0034e-11	3.5499e-87	3.5499e-87	7.83	12	0.702888
$PM2$	3	0.4434	7.9843e-11	1.7686e-87	1.7686e-87	7.86	12	0.787046
$PM3$	3	0.4434	7.9653e-11	3.2750e-88	3.2750e-88	7.94	12	0.759142
$f_4, x_0 = 1.5$								
NM	9	0.7194	0.2927	0.0727	2.7867e-74	2.00	18	1.335429
LWM	4	1.0743	0.0157	3.3096e-14	1.4980e-107	7.99	16	0.873192
SAM	4	1.0905	4.9099e-04	7.3633e-26	1.8691e-200	8.00	16	0.879429
$CFGT$	4	1.0899	8.8191e-05	1.4106e-35	5.5201e-282	8.00	16	0.936900
CTV	4	1.0943	0.0043	2.2678e-19	1.2146e-149	8.00	16	0.907344
NCS	4	1.0814	0.0086	9.2312e-17	1.8264e-128	7.99	16	0.902067
$PM1$	4	1.0848	0.0052	4.2524e-19	9.2702e-148	7.99	16	0.807937
$PM2$	4	1.0879	0.0021	1.0258e-26	1.9657e-214	8.05	16	0.878774
$PM3$	4	1.0899	1.5396e-04	2.7156e-31	2.5507e-245	7.99	16	0.883730

Table 1. Numerical results for test functions.

The path function $y = P(x)$ that describes the motion of the projectile is given by

$$P(x) = h + x \tan\theta - \frac{gx^2}{2v^2} \sec^2\theta \tag{21}$$

When the projectile hits the hill, there is a value of x for which $P(x) = \omega(x)$ for each value of x. We wish to find the value of θ that maximize x.

$$\omega(x) = P(x) = h + x \tan\theta - \frac{gx^2}{2v^2} \sec^2\theta \tag{22}$$

Differentiating eq. (22) implicitly w.r.t. θ, we have

$$\omega'(x)\frac{dx}{d\theta} = x \sec^2\theta + \frac{dx}{d\theta}\tan\theta - \frac{g}{v^2}\left(x^2 \sec^2\theta \tan\theta + x\frac{dx}{d\theta}\sec^2\theta\right) \tag{23}$$

Setting $\frac{dx}{d\theta} = 0$ in eq. (23), we have

$$x_m = \frac{v^2}{g}\cot\theta_m \tag{24}$$

or

$$\theta_m = \arctan\left(\frac{v^2}{g\, x_m}\right) \tag{25}$$

An enveloping parabola is a path that encloses and intersects all possible paths. Henelsmith (Henelsmith 2016) derived an enveloping parabola by maximizing the height of the projectile for a given horizontal distance x, which will give the path that encloses all possible paths. Let $w = \tan\theta$, then eq. (21) becomes

$$y = P(x) = h + xw - \frac{gx^2}{2v^2}(1 + w^2) \tag{26}$$

Differentiating eq. (26) w.r.t. w and setting $y' = 0$, Henelsmith obtained

$$\begin{aligned} y' &= x - \frac{xg^2}{v^2}(w) = 0 \\ w &= \frac{v^2}{g\, x} \end{aligned} \tag{27}$$

so that the enveloping parabola defined by

$$y_m = \rho(x) = h + \frac{v^2}{2g} - \frac{gx^2}{2v^2} \tag{28}$$

The solution to the projectile problem requires first finding x_m which satisfies $\rho(x) = \omega(x)$ and solving for θ_m in eq. (25) because we want to find the point at which the enveloping parabola ρ intersects the impact function ω, and then find θ that corresponds to this point on the enveloping parabola. We choose a linear impact function $\omega(x) = 0.4x$ with $h = 10$ and $v = 20$. We let $g = 9.8$. Then we apply our I.F.s starting from $x_0 = 30$ to solve the non-linear equation

$$f(x) = \rho(x) - \omega(x) = h + \frac{v^2}{2g} - \frac{gx^2}{2v^2} - 0.4x$$

whose root is given by $x_m = 36.102990117.....$ and

$$\theta_m = \arctan\left(\frac{v^2}{g\, x_m}\right) = 48.5^\circ.$$

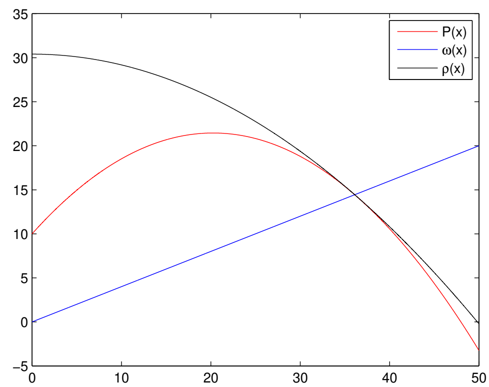

Figure 1. The enveloping parabola with linear impact function.

IF	N	error	$cpu\ time(s)$	ρ
NM	7	4.3980e-76	1.074036	1.99
LWM	3	1.5610e-66	0.658235	8.03
SAM	3	1.2092e-61	0.754623	8.06
$CFGT$	3	3.3018e-89	0.731083	9.03
CTV	3	3.5871e-73	0.689627	8.02
NCS	3	3.0839e-70	0.714955	8.03
$PM1$	3	4.3980e-76	0.683299	8.02
$PM2$	3	4.1159e-112	0.639101	10.03
$PM3$	3	2.3793e-74	0.673341	8.05

Table 2. Numerical results on projectile motion problem

Figure 1 shows the intersection of the path function, the enveloping parabola and the linear impact function for this application. Table 2 shows that proposed method $PM2$ is converges less number of iteration with least error than other compared methods. All the methods are agrees computational order of convergence with their theoretical order of convergence but $PM2$ having highest 10.03. Hence, the proposed method $PM2$ can be considered competent enough to existing other compared equivalent methods.

Example 2: We consider the following Planck's radiation law problem found in (Bradie 2006):

$$\varphi(\lambda) = \frac{8\pi ch\lambda^{-5}}{e^{ch/\lambda kT} - 1}, \tag{29}$$

which calculates the energy density within an isothermal blackbody. Here, λ is the wavelength of the radiation, T is the absolute temperature of the blackbody, k is Boltzmann's constant, h is the Planck's constant and c is the speed of light. Suppose, we would like to determine wavelength λ which corresponds to maximum energy density $\varphi(\lambda)$. From (29), we get

$$\varphi'(\lambda) = \left(\frac{8\pi ch\lambda^{-6}}{e^{ch/\lambda kT} - 1}\right)\left(\frac{(ch/\lambda kT)e^{ch/\lambda kT}}{e^{ch/\lambda kT} - 1} - 5\right) = A \cdot B.$$

It can be checked that a maxima for φ occurs when $B = 0$, that is, when

$$\left(\frac{(ch/\lambda kT)e^{ch/\lambda kT}}{e^{ch/\lambda kT} - 1}\right) = 5.$$

Here putting $x = ch/\lambda kT$, the above equation becomes

$$1 - \frac{x}{5} = e^{-x}. \tag{30}$$

Define

$$f(x) = e^{-x} - 1 + \frac{x}{5}. \tag{31}$$

The aim is to find a root of the equation $f(x) = 0$. Obviously, one of the root $x = 0$ is not taken for discussion. As argued in (Bradie 2006), the left-hand side of (30) is zero for $x = 5$ and $e^{-5} \approx 6.74 \times 10^{-3}$. Hence, it is expected that another root of the equation $f(x) = 0$ might occur near $x = 5$. The approximate root of the equation (31) is given by $x^* \approx 4.96511423174427630369$ with $x_0 = 3$. Consequently, the wavelength of radiation (λ) corresponding to which the energy density is maximum is approximated as

$$\lambda \approx \frac{ch}{(kT)4.96511423174427630369}.$$

IF	N	error	$cpu\ time(s)$	ρ
NM	7	1.8205e-70	1.043458	2.00
LWM	4	3.1188e-268	0.872883	7.99
SAM	4	1.9335e-298	0.869115	8.00
$CFGT$	4	0	0.937835	7.99
CTV	4	5.8673e-282	0.890461	8.00
NCS	4	5.9695e-314	0.888809	7.99
$PM1$	4	2.5197e-322	0.813324	8.00
$PM2$	4	0	0.866063	8.00
$PM3$	4	0	0.857505	7.99

Table 3. Numerical results on Planck's radiation law problem

Table 3 shows that the method $CFGT$, $PM3$ and $PM4$ are converges less number of iteration with least error than other compared methods. All the methods are agrees computational order of convergence with their theoretical order of convergence.

Example 3: In the study of the multi-factor effect, the trajectory of an electron in the air gap between two parallel plates is given by

$$x(t) = x_0 + (v_0 + e\frac{E}{m\omega} sin\, \omega t_0 + \alpha)(t - t_0) + e\frac{E_0}{m\omega^2}(cos(\omega t + \alpha)), \tag{32}$$

where e and m are charge and the mass of the electron at rest, x_0 and v_0 are the position and velocity of the electron at time t_0 and $E_0\ sin(\omega t + \alpha)$ is the RF electric field between the plates. We choose the particular parameters in the equation (33) in order to deal with a simpler expression, which is defined as follows:

$$f(x) = x - \frac{1}{2}cosx + \frac{\pi}{4}. \tag{33}$$

The required zero of the above function is $x^* = 0.309093271541\ldots$, with $x_0 = -0.9$. Table 4 shows that the method $PM2$ is converges less number of iteration with least error than other compared methods. All the methods are agrees computational order of convergence with their theoretical order of convergence.

IF	N	error	$cpu\ time(s)$	ρ
NM	8	9.4025e-98	1.226429	1.99
LWM	3	4.2378e-52	0.774762	7.84
SAM	4	2.1252e-259	0.902507	8.00
$CFGT$	4	0	1.041553	7.99
CTV	3	8.7703e-53	0.782078	7.92
NCS	4	0	0.976453	7.99
$PM1$	3	2.5582e-56	0.714148	7.97
$PM2$	4	0	1.014608	7.99
$PM3$	4	0	1.003232	7.99

Table 4. Numerical results on trajectory of an electron problem

Example 4: Van der Waal's equation of state

$$(P + \frac{an^2}{V^2})(V - nb) = nRT. \tag{34}$$

The determination of the volume V of the gas in terms of the remaining parameters requires the solution of a nonlinear equation in V.

$$PV^3 - (nbP + nRT)V^2 + an^2V - an^2b = 0. \tag{35}$$

Given the constants a and b of a particular gas, one can find values for n, P and T, such that this equation has three roots. By using the specific values, one can obtain the following nonlinear function

$$f(x) = 0.986\ x^3 - 5.181\ x^2 + 9.067\ x - 5.289 \tag{36}$$

The required zero of the above function is $x^* = 1.929846242847\ldots$, with $x_0 = 2.5$.

IF	N	error	$cpu\ time(s)$	ρ
NM	11	1.3111e-64	1.625181	1.99
LWM	5	1.4477e-173	1.060197	7.99
SAM	5	2.6242e-303	1.049326	8.00
$CFGT$	31	3.7158e-51	6.449081	0.99
CTV	4	2.0395e-85	0.929204	7.98
NCS	5	5.9980e-213	1.079848	7.99
$PM1$	5	3.6290e-292	1.187237	7.99
$PM2$	4	1.0795e-68	0.981485	7.72
$PM3$	4	3.3998e-98	0.977020	7.98

Table 5. Numerical results on Van der Waal's equation

Table 5 shows that the method $PM3$ is converges less number of iteration with least error than other compared methods. All the methods are agrees computational order of convergence with their theoretical order of convergence.

5 Basins of Attraction

Iterative scheme gives information about convergence and stability of the method by studying basins of attraction of the rational function. The basic definitions and dynamical concepts of rational function which can found in (Amat, Busquier, and Plaza 2004b) and (Scott, Neta, and Chun 2011). Let us consider a square region $\mathbb{R} \times \mathbb{R} = [-3, 3] \times [-3, 3]$ of 300×300 grid points and we apply iterative methods starting in every grid point $z^{(0)}$ in the square. The iterative method attempts a zero z_j^* of the equation with a condition $|f(z^{(k)})| < 1e-4$ and a maximum of 100 iterations, we conclude that $z^{(0)}$ is in the basin of attraction of this zero. If the iterative method starting in $z^{(0)}$ reaches a zero in N iterations ($N \leq 100$), then we mark this point $z^{(0)}$ with colors if $|z^{(N)} - z_j^*| < 1e-4$. If $N > 50$, we conclude that the initial point has diverged and we assign a dark blue color. We describe the basins of attraction for finding complex roots of polynomials $p_1(z) = z^2 - 1$, $p_2(z) = z^3 - 1$ and $p_3(z) = z^4 - 1$.

In Figures 2-4, we have given the basins attraction for new methods and some existing methods. We conform that a point z_0 containing to the Julia set whenever the dynamics in a neighborhood of point displays sensitive on the conditions based. Therefore, nearby initial conditions leading to the slightly different behavior later in some number of iterations. Therefore, some compared methods are obtaining many divergent initial points. The boundaries of the basins of attraction are Julia set of the iteration function.

6 Concluding Remarks

In this work, we have developed a family of optimal eighth order iterative methods for solving nonlinear equations. The methods requires four functions evaluations reaching order of convergence is eight. In the sense of convergence analysis and numerical examples, the Kung-Traub's conjecture is satisfied. We have tested some examples using the proposed schemes and some known schemes, which illustrate the superiority of the proposed methods. The numerical experiments suggests that the new methods would be valuable alternative for solving nonlinear equations. We also solve some real world applications of different nature to show the effectiveness of the proposed schemes. Further, investigation has been made on the complex plane for such methods to reveal their basins of attraction for solving nonlinear equations by presenting their corresponding fractals. Further, we are investigating the proposed scheme to develop optimal methods of arbitrarily high order 2^{d-1} with multi-step Newton's method.

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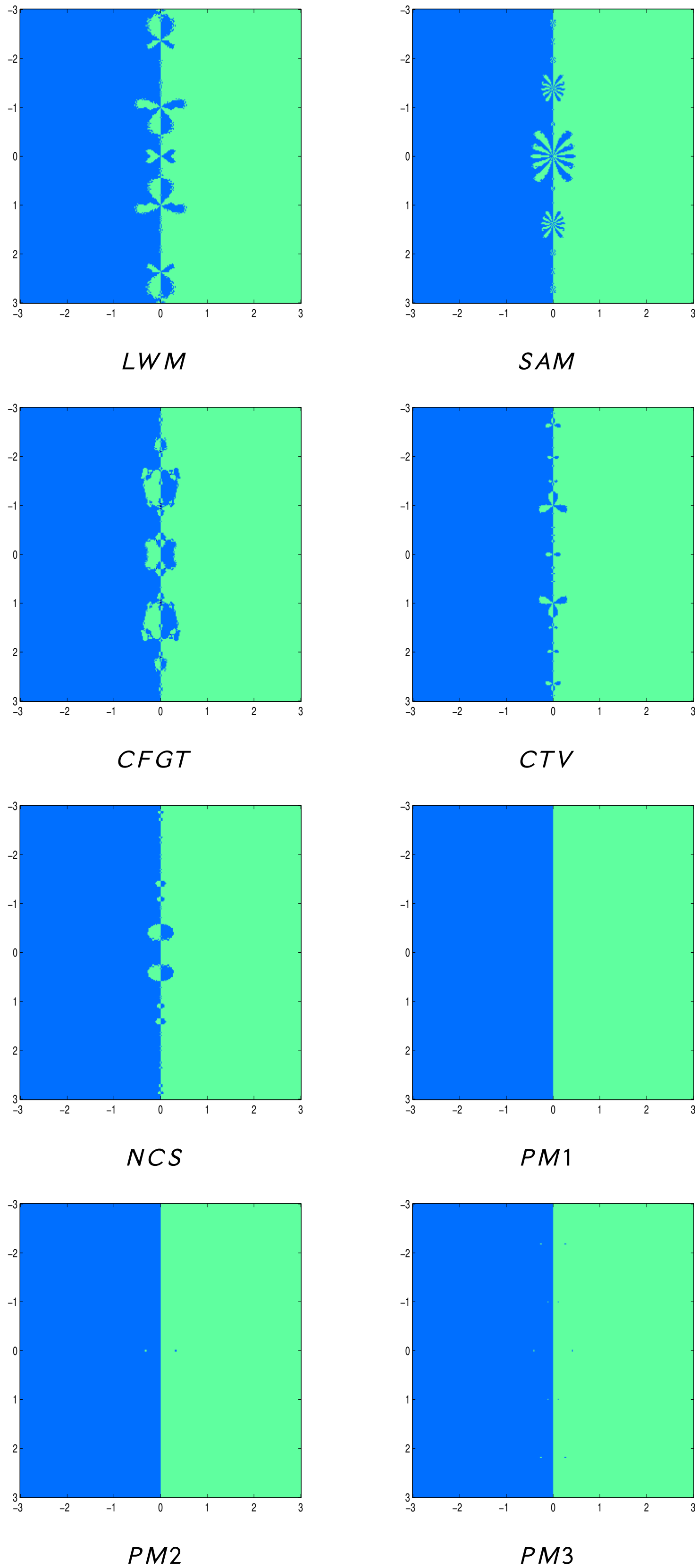

Figure 2. Basins of attraction for $p_1(z) = z^2 - 1$

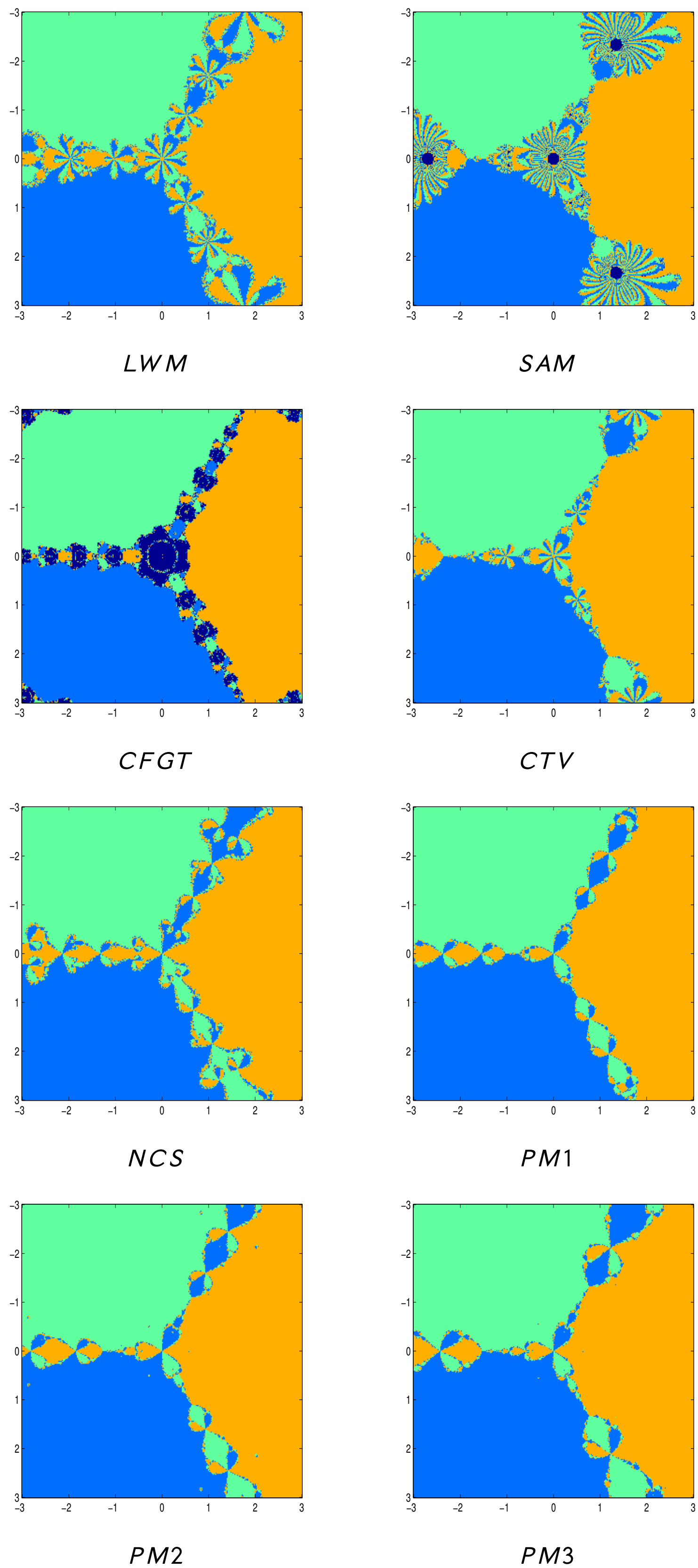

Figure 3. Basins of attraction for $p_2(z) = z^3 - 1$

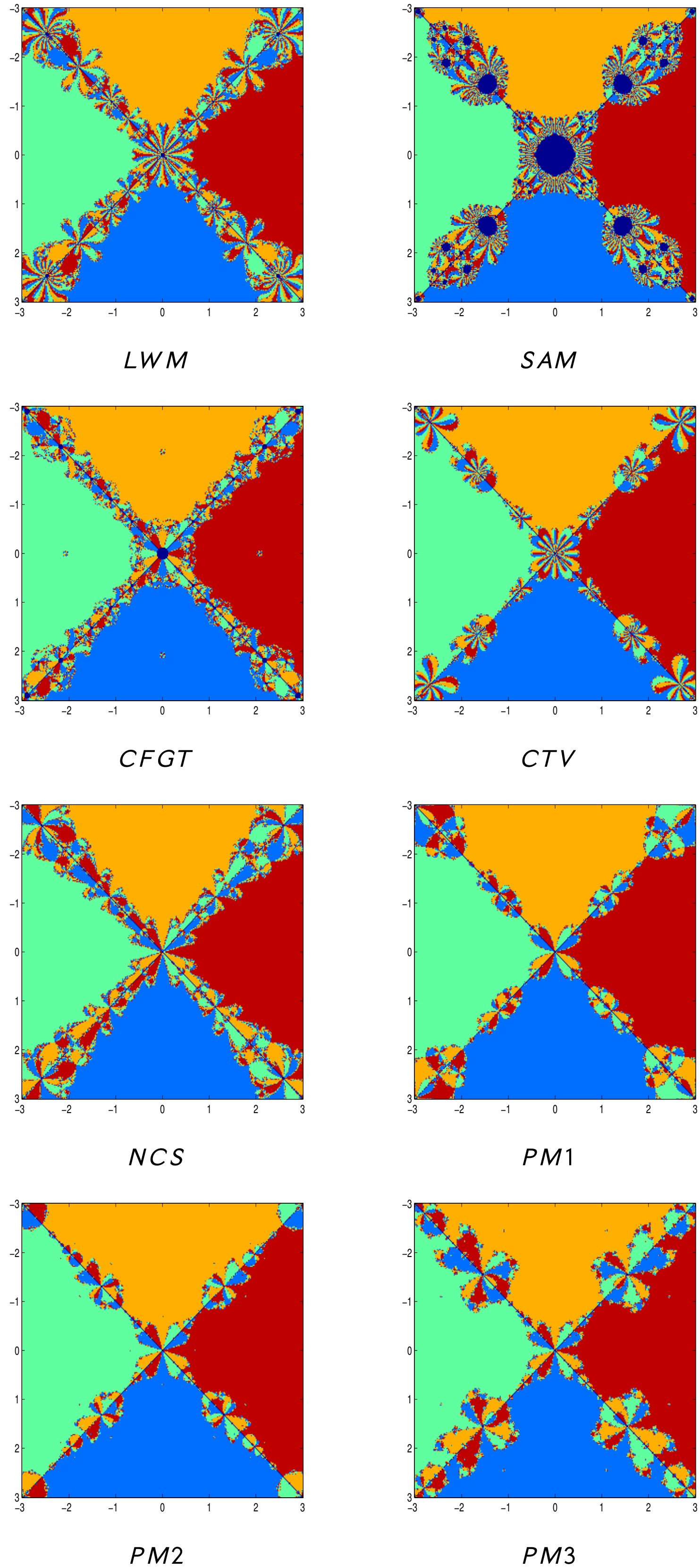

Figure 4. Basins of attraction for $p_3(z) = z^4 - 1$

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