*Computational Methods, 3(2):35–43, July-December: 2026.*
*DOI: https://doi.org/10.58614/cm325*

**Computational Methods**
*ISSN (Online): 3048-7447*

*Journal homepage: https://www.jpub.org/journal-details.php?journal-id=43*

***Research Article***

# Geometric Properties of Clausen's Hypergeometric Functions on Janowski Type

**D. Elumalai[1*] and R. Ganapathy Raman[2]**

[1]*Department of Mathematics, Aringar Anna Institute of Science and Technology, Sriperumputhur, 600087, Tamil Nadu, India.*
[2]*PG and Research Department of Mathematics, Pachiyappa's College, Chennai, 600030, Tamil Nadu, India.*
[*]*Corresponding author: duraisamyelumalai@gmail.com*

### Article Info

***Keywords:*** *Clausen's Hypergeometric Function, Univalent Functions, Starlike Functions, Convex Functions and Close-to-Convex functions.*

***Received:*** *02.07.2026;*
***Accepted:*** *18.09.2026;*
***Published:*** *27.09.2026*



### Abstract

In this paper, an operator $\mathscr{I}_{d,e}^{a,b,c}(f)(z)$ involving Clausen's Hypergeometric Function by means of the Hadamard Product is considered. The geometric properties of this operator $\mathscr{I}_{d,e}^{a,b,c}(f)(z)$ are obtained based on its Taylor's coefficient for various subclasses of univalent functions.

## 1. Introduction

Let $\mathscr{A}$ represent the family of functions $f$ with the specified form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are normalized and analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ of the complex plane.

Let $\mathscr{S}$, $\mathscr{S}^*$, $\mathscr{C}$, $\mathscr{K}$ and $\mathscr{P}$ denote the classes of univalent, starlike, convex, close-to-convex functions and function with positive real part, respectively. (For more details about the properties of univalent, starlike, and convexity functions, see [1, 2]).
The concept of uniformly convex and uniformly starlike functions was introduced by Goodman[3, 4] and denoted as $\mathscr{UCV}$ and $\mathscr{UST}$, respectively. Subsequently, Rønning [5] and Ma and Minda [6] independently provided the analytic characterization of the class $\mathscr{UCV}$ in one variable, which is as follows: $f \in \mathscr{UCV}$ if and only if $Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|$.

The subclass $\mathscr{S}_p$ of starlike functions, introduced by Rønning [5] is defined as

$$\mathscr{S}_p = \{F \in \mathscr{S}^* | F(z) = zf'(z), f(z) \in \mathscr{UCV}\}. \tag{2}$$

It is clear that $f(z) \in \mathscr{S}_p$ if and only if

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < Re\left(\frac{zf'(z)}{f(z)}\right), z \in \mathbb{D}. \tag{3}$$

We observe that $\mathscr{S}_p$ is the class of functions for which the range of $zf'(z)/f(z)$, with $z \in \mathbb{D}$ is the region $\Omega$, where $\Omega = \{w : Re(w) > |w-1|\}$. Notably, $\Omega$ represents the interior of a parabola in the right half plane that is symmetric with respect to the real axis and has its vertex at $\left(\frac{1}{2}, 0\right)$. It is well- known that the function

$$p_1(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 \tag{4}$$

maps the unit disc $\mathbb{D}$ onto the parabolic region $\Omega$, and therefore, it belongs to $\mathscr{S}_p$.

In 1999, Kanas and Wisniowska [7, 8] generalized the above parabolic domain and introduced the conic domain $\Omega_k, k \geq 0$ and studied it comprehensively. This domain is defined as

$$\Omega_k = \{w : Re(w) > k|w-1|\}.$$

This domain represents the right half plane for $k = 0$, hyperbolic regions (right branch) when $0 < k < 1$, a parabolic region for $k = 1$ and elliptic regions when $k > 1$.
The functions which play the role of extremal functions for these conic regions are given as:

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0 \\ 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1 \\ 1 + \frac{2}{1-k^2}\sinh^2\left(\left(\frac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right), & 0 < k < 1 \\ 1 + \frac{1}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}\right) + \frac{1}{k^2-1}, & k > 1 \end{cases} \tag{5}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}, t \in (0,1), z \in \mathbb{D}$ and $z$ is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is the Legendreís complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$, for more detail, [7, 8].
Using subordination criterion, Janowski [9] introduced another class $\mathscr{P}[A,B]$ a subclass of $\mathscr{P}$ and studied its properties.

For $-1 \leq B < A \leq 1$, let

$$\mathscr{P}[A,B] = \left\{f \in \mathscr{A} : f \prec \frac{1+Az}{1+Bz}, z \in \mathbb{D}\right\}.$$

For $A = 1, B = -1$, a well known family $\mathscr{P}$ functions with positive real part.
A function [10] $p(z)$ is said to be in the class $P_1[A,B]$;if and only if

$$p(z) \prec \frac{(A+1)p_1(z) - (A-1)}{(B+1)p_1(z) - (B-1)} \tag{6}$$

where $p_1(z)$ is defined by (4) and $-1 \leq B < A \leq 1$.
A function [10] $f(z) \in \mathscr{A}$ is said to be in the class $\mathscr{UCV}[A,B]$ $-1 \leq B < A \leq 1$ if and only if ,

$$\mathrm{Re}\left(\frac{(B-1)\frac{(zf'(z))'}{f'(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)} - (A+1)}\right) > \left|\frac{(B-1)\frac{((zf'(z))'}{f'(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)} - (A+1)} - 1\right| \tag{7}$$

A function [10] $f(z) \in \mathscr{A}$ is said to be in the class $\mathscr{S}_p[A,B]$ $-1 \leq B < A \leq 1$ if and only if ,

$$\mathrm{Re}\left(\frac{(B-1)\frac{zf'(z)}{f(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f(z)} - (A+1)}\right) > \left|\frac{(B-1)\frac{zf'(z)}{f(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f(z)} - (A+1)} - 1\right| \tag{8}$$

The following lemmas gives the sufficient condition for a function $f$ of the form (1) to belong to the class $\mathscr{UCV}[A,B]$ and $\mathscr{S}_p[A,B]$ respectively.

**Lemma 1.1.** *[10] A function $f(z) \in \mathscr{A}$ and of the form (1) is in the class $\mathscr{UCV}[A,B]$ if it satisfies the condition*

$$\sum_{n=2}^{\infty} n\left[4(n-1)+|n(B+1)-(A+1)|\right]|A_n| \leq |B-A|, \tag{9}$$

*where $-1 \leq B < A \leq 1$.*

**Lemma 1.2.** *[10] A function $f(z) \in \mathscr{A}$ and of the form (1) is in the class $\mathscr{S}_p[A,B]$ if it satisfies the condition*

$$\sum_{n=2}^{\infty} \left[4(n-1)+|n(B+1)-(A+1)|\right]|A_n| \leq |B-A|. \tag{10}$$

*where $-1 \leq B < A \leq 1$.*

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be analytic in $\mathbb{D}$. Then, the Hadamard product or convolution of $f(z)$ and $g(z)$ is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

For [11] $\beta < 1$, let

$$\mathscr{R}(\beta) = \{f \in \mathscr{A} : \exists\, \eta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) | \operatorname{Re}\, [e^{i\eta}(f'(z)-\beta)] > 0, z \in \mathbb{D}\}.$$

It is clear that $\mathscr{R}(\beta) \subset \mathscr{S}$ when $\beta \geq 0$ and for each $\beta < 0$, $\mathscr{R}(\beta)$ also includes non-univalent functions. This class has been extensively used to investigate various integral transforms. For example, refer to [12, 13] and the references cited therein.

Suppose that $f(z)$ is of the form (1) is in class $\mathscr{R}(\beta)$, then by [11], we have

$$|a_n| \leq \frac{2(1-\beta)}{n}, \; n \geq 2. \tag{11}$$

For any non-zero complex variable $a$, the Pochhammer symbol (or shifted factorial) is defined as

$$(a)_0 = 1, \text{ and } (a)_n = a(a+1)\cdots(a+n-1), \text{ for } n = 1,2,3,\cdots.$$

In terms of Euler gamma function, the Pochhammer symbol can be written as

$$(a)_n = \frac{\Gamma(n+a)}{\Gamma(a)}, \quad n = 0,1,2,\cdots$$

where, $a$ is neither zero, nor a negative integer.

The Clausen's hypergeometric function ${}_3F_2(a,b,c;d,e;z)$ is defined as

$${}_3F_2(a,b,c;d,e;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n(1)_n} z^n, \; a,b,c,d,e \in \mathbb{C}, \tag{12}$$

provided $d$, $e \neq 0,-1,-2,-3\cdots$, which is an analytic function in unit disc $\mathbb{D}$.

Following the proof of the Bieberbach conjecture, many researches in the field of function theory recognized the significance of special function, in particular, the hypergeometric function, in relation to geometric function theory. Subsequently, numerous results on the Gaussian Hypergeometric function [12–24] have emerged in connection with various subclasses of univalent functions.

In recent years, much research has focus on generalized hypergeometric functions and their connections to various subclasses of univalent functions.

Recently, Prabhakaran and Chandrasekar [25] introduced a new integral operation involving the Clausen hypergeometric function. Using this convolution operation, they have established several new results related to generalized hypergeometric functions.

Building on their work, we also examine the geometric properties of various subclasses of univalent functions, making our result more general.

For $f \in \mathscr{A}$, recall the operator $\mathscr{I}_{d,e}^{a,b,c}(f)(z)$

$$\mathscr{I}_{d,e}^{a,b,c}(f)(z) = z\,{}_3F_2(a,b,c;d,e;z) * f(z) = z + \sum_{n=2}^{\infty} A_n z^n \tag{13}$$

with $A_1 = 1$ and for $n > 1$,

$$A_n = \frac{(a)_{n-1}(b)_{n-1}(c)_{n-1}}{(d)_{n-1}(e)_{n-1}(1)_{n-1}} a_n. \tag{14}$$

In the research conducted thus far, no general summation formula has been found that addresses the parameters $a$, $b$, $c$, $d$, and $e$. However, there are a few summation formulas with the restricted coefficients, such as those by Dixon's and Schultz's.

Recently, Miller and Paris [26] derived a summation formula involving $n$. Additionally, Shpot and Srivastava [27] discovered a new summation formula involving both $m$ and $n$. We present the restricted form of their summation formula as follows:

$$ {}_3F_2(a,b,c;b+1,c+1;1) = \frac{bc}{c-b}\Gamma(1-a)\left[\frac{\Gamma(b)}{\Gamma(1-a+b)} - \frac{\Gamma(c)}{\Gamma(1-a+c)}\right], \tag{15} $$

provided that, $a < 2$, $b > a-1$, and $c > a-1$. The equation (6) is derived by putting $n = 1$ in Theorem 1 in Miller and Paris [26] and by putting $m = n = 0$ in the equation (3) in Shpot and Srivastava [27].

The following Lemma is help us to prove our desired results.

**Lemma 1.3.** *[25] Let $a$, $b$, $c > 0$ and $c \neq b$. Then we have the following:*

1. *For $a < \min\{1, b+1, c+1\}$, we have*

$$ \sum_{n=0}^{\infty} \frac{(n+1)(a)_n\,(b)_n\,(c)_n}{(b+1)_n\,(c+1)_n\,(1)_n} = \frac{bc\Gamma(1-a)}{c-b}\left[\frac{(1-b)\Gamma(b)}{\Gamma(1-a+b)} - \frac{(1-c)\Gamma(c)}{\Gamma(1-a+c)}\right]. $$

2. *For $a < \min\{1, b+1, c+1\}$, we have*

$$ \sum_{n=0}^{\infty} \frac{(n+1)^2(a)_n\,(b)_n\,(c)_n}{(b+1)_n\,(c+1)_n\,(1)_n} = \frac{bc\Gamma(1-a)}{c-b}\left[\frac{(1-b)^2\Gamma(b)}{\Gamma(1-a+b)} - \frac{(1-c)^2\Gamma(c)}{\Gamma(1-a+c)}\right]. $$

3. *For $a < \min\{1, b+1, c+1\}$, we have*

$$ \sum_{n=0}^{\infty} \frac{(n+1)^3(a)_n\,(b)_n\,(c)_n}{(b+1)_n\,(c+1)_n\,(1)_n} = \frac{bc\Gamma(1-a)}{c-b}\left[\frac{[(1-b)^3]\Gamma(b)}{\Gamma(1-a+b)} - \frac{[(1-c)^3]\Gamma(c)}{\Gamma(1-a+c)}\right]. $$

4. *For $a \neq 1, b \neq 1,\ c \neq 1$ and $a < \min\{1, b+1, c+1\}$, we have*
$\displaystyle\sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\,(c)_n}{(b+1)_n\,(c+1)_n\,(1)_{(n+1)}}$

$$ = \left(\frac{bc}{(a-1)(b-1)(c-1)}\right)\left[\frac{\Gamma(2-a)}{c-b}\left(\frac{(c-1)\Gamma(b)}{\Gamma(1-a+b)} - \frac{(b-1)\Gamma(c)}{\Gamma(1-a+c)}\right) - 1\right]. $$

## 2. Main Theorems and Proofs

**Theorem 2.1.** *Let $a \in \mathbb{C}\backslash\{0\},\ b > |a|-1$ and $c > |a|-1$. A sufficient condition for the function $z\,{}_3F_2(a,b,c;b+1,c+1;z)$ to belong to the class $\mathscr{UCV}[A,B]$ is that*

$$ \frac{bc\Gamma(1-|a|)}{c-b}\left[[(1-b)[(1-b)(4+|B+1|)+|A+1|-4]]\frac{\Gamma(b)}{\Gamma(1-|a|+b)} \right. $$
$$ \left. -[(1-c)[(1-c)(4+|B+1|)+|A+1|-4]]\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\right] \leq 2|B-A|. $$

*Proof.* Let $a \in \mathbb{C}\backslash\{0\},\ b > |a|-1,\ c > |a|-1$.

Let $f(z) = z\,{}_3F_2(a,b,c;b+1,c+1;z)$. Then, by (9), it is enough to show that

$$ T = \sum_{n=2}^{\infty} n\Big(4(n-1) + |n(B+1)-(A+1)|\Big)\,|A_n| \leq |B-A|. $$

where $A_n$ is given by (14). Using the fact $|(a)_n| \leq (|a|)_n$,

$$\begin{aligned}
T &\leq \sum_{n=2}^{\infty} n\left(4(n-1)+|n(B+1)-(A+1)|\right)\left(\frac{(|a|)_{n-1}(b)_{n-1}(c)_{n-1}}{(b+1)_{n-1}(c+1)_{n-1}(1)_{n-1}}\right) \\
&= \sum_{n=1}^{\infty}(n+1)\left(4n+|(n+1)(B+1)-(A+1)|\right)\left(\frac{(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_n}\right) \\
&= \sum_{n=1}^{\infty}(n+1)\left(4(n+1-1)+|(n+1)(B+1)-(A+1)|\right)\left(\frac{(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_n}\right) \\
&= \sum_{n=0}^{\infty}(n+1)\left(4(n+1-1)+|(n+1)(B+1)-(A+1)|\right)\left(\frac{(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_n}\right) \\
&\quad -|B-A| \\
&= 4\sum_{n=0}^{\infty}(n+1)^2\left(\frac{(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_n}\right)-4\sum_{n=0}^{\infty}(n+1)\left(\frac{(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_n}\right) \\
&\quad +|(B+1)|\sum_{n=0}^{\infty}\left(\frac{(n+1)^2\,(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_n}\right)+|(A+1)|\sum_{n=0}^{\infty}\left(\frac{(n+1)\,(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_n}\right) \\
&\quad -|B-A|
\end{aligned}$$

Using the formula (15) and results (1) and (2) of Lemma 1.3 in the aforementioned equation, we find that

$$\begin{aligned}
T &\leq 4\frac{bc\Gamma(1-|a|)}{c-b}\left[\frac{(1-b)^2\Gamma(b)}{\Gamma(1-|a|+b)}-\frac{(1-c)^2\Gamma(c)}{\Gamma(1-|a|+c)}\right] \\
&\quad -4\frac{bc\Gamma(1-|a|)}{c-b}\left[\frac{(1-b)\Gamma(b)}{\Gamma(1-|a|+b)}-\frac{(1-c)\Gamma(c)}{\Gamma(1-|a|+c)}\right] \\
&\quad +|B+1|\frac{bc\Gamma(1-|a|)}{c-b}\left[\frac{(1-b)^2\Gamma(b)}{\Gamma(1-|a|+b)}-\frac{(1-c)^2\Gamma(c)}{\Gamma(1-|a|+c)}\right] \\
&\quad +|A+1|\frac{bc\Gamma(1-|a|)}{c-b}\left[\frac{(1-b)\Gamma(b)}{\Gamma(1-|a|+b)}-\frac{(1-c)\Gamma(c)}{\Gamma(1-|a|+c)}\right]-|B-A| \\
&\leq \frac{bc\Gamma(1-|a|)}{c-b}\Bigg[[(1-b)[4(1-b)-4+(1-b)|B+1|+|A+1|]]\frac{\Gamma(b)}{\Gamma(1-|a|+b)} \\
&\quad -[(1-c)[4(1-c)-4+(1-c)|B+1|+|A+1|]]\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\Bigg]-|B-A| \\
&= \frac{bc\Gamma(1-|a|)}{c-b}\Bigg[[(1-b)[(1-b)(4+|B+1|)+|A+1|-4]]\frac{\Gamma(b)}{\Gamma(1-|a|+b)} \\
&\quad -[(1-c)[(1-c)(4+|B+1|)+|A+1|-4]]\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\Bigg]-|B-A|.
\end{aligned}$$

Because of (9), the above expression is bounded above by $|B-A|$ , and hence,

$$\begin{aligned}
&\frac{bc\Gamma(1-|a|)}{c-b}\Bigg[[(1-b)[(1-b)(4+|B+1|)+|A+1|-4]]\frac{\Gamma(b)}{\Gamma(1-|a|+b)} \\
&-[(1-c)[(1-c)(4+|B+1|)+|A+1|-4]]\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\Bigg]-|B-A|\leq|B-A|.
\end{aligned}$$

Therefore, $z_3F_2(a,b,c;b+1,c+1;z)$ belongs to the class $\mathcal{UCV}[A,B]$. □

**Theorem 2.2.** *Let $a\in\mathbb{C}\backslash\{0\},\ b>|a|-1$ and $c>|a|-1$,*

$$\begin{aligned}
&\frac{bc\Gamma(1-|a|)}{c-b}\Bigg[[(1-b)(4+|B+1|)+|A+1|-4]\frac{\Gamma(b)}{\Gamma(1-|a|+b)} \\
&-[(1-c)(4+|B+1|)+|A+1|-4]\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\Bigg]\leq|B-A|\left(1+\frac{1}{2(1-\beta)}\right).
\end{aligned}$$

*Then $\mathscr{I}^{a,b,c}_{b+1,c+1}(f)(z)$ maps $\mathscr{R}(\beta)$ into $\mathcal{UCV}[A,B]$.*

*Proof.* Let $a\in\mathbb{C}\backslash\{0\},\ b>|a|-1,\ c>|a|-1$.

Let $f(z) = z\,{}_3F_2(a,b,c;b+1,c+1;z)$. Then, by (9), it is enough to show that

$$T = \sum_{n=2}^{\infty} n\left(4(n-1)+|n(B+1)-(A+1)|\right)|\boldsymbol{A}_n| \le |B-A|.$$

where $A_n$ is given by (14). Using the fact $|(a)_n| \le (|a|)_n$, and (11) in the above equation it is found that

$$\begin{aligned}
T &\le \sum_{n=2}^{\infty} n\left(4(n-1)+|n(B+1)-(A+1)|\right)\left(\frac{(|a|)_{n-1}(b)_{n-1}(c)_{n-1}}{(b+1)_{n-1}(c+1)_{n-1}(1)_{n-1}}\right)(a)_n \\
&\le 2(1-\beta)\sum_{n=2}^{\infty}\left(4(n-1)+|n(B+1)-(A+1)|\right)\left(\frac{(|a|)_{n-1}(b)_{n-1}(c)_{n-1}}{(b+1)_{n-1}(c+1)_{n-1}(1)_{n-1}}\right) \\
&= 2(1-\beta)\sum_{n=1}^{\infty}\left(4n+|(n+1)(B+1)-(A+1)|\right)\left(\frac{(|a|)_{n}(b)_{n}(c)_{n}}{(b+1)_{n}(c+1)_{n}(1)_{n}}\right) \\
&= 2(1-\beta)\sum_{n=1}^{\infty}\left(4(n+1-1)+|(n+1)(B+1)-(A+1)|\right)\left(\frac{(|a|)_{n}(b)_{n}(c)_{n}}{(b+1)_{n}(c+1)_{n}(1)_{n}}\right) \\
&= 2(1-\beta)\sum_{n=0}^{\infty}\left(4(n+1-1)+|(n+1)(B+1)-(A+1)|\right)\left(\frac{(|a|)_{n}(b)_{n}(c)_{n}}{(b+1)_{n}(c+1)_{n}(1)_{n}}\right) \\
&\quad -|B-A| \\
&= 2(1-\beta)\left[4\sum_{n=0}^{\infty}(n+1)\left(\frac{(|a|)_{n}(b)_{n}(c)_{n}}{(b+1)_{n}(c+1)_{n}(1)_{n}}\right)-4\sum_{n=0}^{\infty}\left(\frac{(|a|)_{n}(b)_{n}(c)_{n}}{(b+1)_{n}(c+1)_{n}(1)_{n}}\right)\right. \\
&\quad +|(B+1)|\sum_{n=0}^{\infty}\left(\frac{(n+1)\,(|a|)_{n}(b)_{n}(c)_{n}}{(b+1)_{n}(c+1)_{n}(1)_{n}}\right)+|(A+1)|\sum_{n=0}^{\infty}\left(\frac{(|a|)_{n}(b)_{n}(c)_{n}}{(b+1)_{n}(c+1)_{n}(1)_{n}}\right) \\
&\quad \left.-|B-A|\right]
\end{aligned}$$

Using the formula (15) and results (1) of Lemma 1.3 in the aforementioned equation, we find that

$$\begin{aligned}
T &\le 2(1-\beta)\left[4\frac{bc\Gamma(1-|a|)}{c-b}\left[\frac{(1-b)\Gamma(b)}{\Gamma(1-|a|+b)}-\frac{(1-c)\Gamma(c)}{\Gamma(1-|a|+c)}\right]\right. \\
&\quad -4\frac{bc\Gamma(1-|a|)}{c-b}\left[\frac{\Gamma(b)}{\Gamma(1-|a|+b)}-\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\right] \\
&\quad +|B+1|\frac{bc\Gamma(1-|a|)}{c-b}\left[\frac{(1-b)\Gamma(b)}{\Gamma(1-|a|+b)}-\frac{(1-c)\Gamma(c)}{\Gamma(1-|a|+c)}\right] \\
&\quad \left.+|A+1|\frac{bc\Gamma(1-|a|)}{c-b}\left[\frac{\Gamma(b)}{\Gamma(1-|a|+b)}-\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\right]-|B-A|\right] \\
&\le 2(1-\beta)\left[\frac{bc\Gamma(1-|a|)}{c-b}\left[[4(1-b)-4+(1-b)|B+1|+|A+1|]\frac{\Gamma(b)}{\Gamma(1-|a|+b)}\right.\right. \\
&\quad \left.\left.-[4(1-c)-4+(1-c)|B+1|+|A+1|]\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\right]-|B-A|\right] \\
&= 2(1-\beta)\left[\frac{bc\Gamma(1-|a|)}{c-b}\left[[(1-b)(4+|B+1|)+|A+1|-4]\frac{\Gamma(b)}{\Gamma(1-|a|+b)}\right.\right. \\
&\quad \left.\left.-[(1-c)(4+|B+1|)+|A+1|-4]\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\right]-|B-A|\right].
\end{aligned}$$

Because of (9), the above expression is bounded above by $|B-A|$ , and hence,

$$\begin{aligned}
&2(1-\beta)\left[\frac{bc\Gamma(1-|a|)}{c-b}\left[[(1-b)(4+|B+1|)+|A+1|-4]\frac{\Gamma(b)}{\Gamma(1-|a|+b)}\right.\right. \\
&\left.\left.-[(1-c)(4+|B+1|)+|A+1|-4]\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\right]-|B-A|\right] \le |B-A|.
\end{aligned}$$

Therefore, $\mathscr{I}^{a,b,c}_{b+1,c+1}(f)(z)$ maps $\mathscr{R}(\beta)$ into $\mathscr{UCV}[A,B]$. □

**Theorem 2.3.** *Let $a \in \mathbb{C}\backslash\{0\}$, $b > |a|-1$ and $c > |a|-1$. A sufficient condition for the function $z\,{}_3F_2(a,b,c;b+1,c+1;z)$ to belong to the class $\mathscr{S}_p[A,B]$ is that*

$$\begin{aligned}
&\frac{bc\Gamma(1-|a|)}{c-b}\left[[(1-b)(4+|B+1|)+|A+1|-4]\frac{\Gamma(b)}{\Gamma(1-|a|+b)}\right. \\
&\left.-[(1-c)(4+|B+1|)+|A+1|-4]\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\right] \le 2|B-A|.
\end{aligned}$$

*Proof.* The proof is similar to Theorem 2.2. So we omit the details. □

**Theorem 2.4.** *Let $a \in \mathbb{C}\backslash\{0\}$, $b > |a|-1$ and $c > |a|-1$.*

$$\frac{bc\Gamma(1-|a|)}{c-b}\left[[(1-b)[(1-b)(4+|B+1|)+|A+1|-4]]\frac{\Gamma(b)}{\Gamma(1-|a|+b)}\right.$$
$$\left.-[(1-c)[(1-c)(4+|B+1|)+|A+1|-4]]\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\right] \leq 2|B-A|.$$

*Then $\mathscr{I}^{a,b,c}_{b+1,c+1}(f)(z)$ maps $S$ into $\mathscr{S}_p[A,B]$.*

*Proof.* The proof is similar to Theorem 2.1. So we omit the details. □

**Theorem 2.5.** *Let $a \in \mathbb{C}\backslash\{0\}$, $b > |a|-1$ and $c > |a|-1$.*

$$\frac{bc\Gamma(1-|a|)}{c-b}\left[\left(4+|B+1|+\frac{4-|A+1|}{(b-1)}\right)\frac{\Gamma(b)}{\Gamma(1-|a|+b)}\right.$$
$$\left.-\left(4+|B+1|+\frac{4-|A+1|}{(c-1)}\right)\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\right]$$
$$+(4-|A+1|)\left(\frac{bc}{(|a|-1)(b-1)(c-1)}\right) \leq |B-A|\left(1+\frac{1}{2(1-\beta)}\right)$$

*Then $\mathscr{I}^{a,b,c}_{b+1,c+1}(f)(z)$ maps $\mathscr{R}(\beta)$ into $\mathscr{S}_p[A,B]$.*

*Proof.* Let $a \in \mathbb{C}\backslash\{0\}$, $b > |a|-1$, $c > |a|-1$.

Let $f(z) = z\,{}_3F_2(a,b,c;b+1,c+1;z)$. Then, by (10), it is enough to show that

$$T = \sum_{n=2}^{\infty}\Big(4(n-1)+|n(B+1)-(A+1)|\Big)|A_n| \leq |B-A|.$$

where $A_n$ is given by (14). Using the fact $|(a)_n| \leq (|a|)_n$, and (11) in the above equation it is found that

$$\begin{aligned}
T &\leq \sum_{n=2}^{\infty}\Big(4(n-1)+|n(B+1)-(A+1)|\Big)\left(\frac{(|a|)_{n-1}(b)_{n-1}(c)_{n-1}}{(b+1)_{n-1}(c+1)_{n-1}(1)_{n-1}}\right)(a)_n\\
&\leq 2(1-\beta)\sum_{n=2}^{\infty}\Big(4(n-1)+|n(B+1)-(A+1)|\Big)\left(\frac{(|a|)_{n-1}(b)_{n-1}(c)_{n-1}}{(b+1)_{n-1}(c+1)_{n-1}(1)_{n-1}\,n}\right)\\
&= 2(1-\beta)\sum_{n=1}^{\infty}\Big(4n+|(n+1)(B+1)-(A+1)|\Big)\left(\frac{(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_{n+1}}\right)\\
&= 2(1-\beta)\sum_{n=1}^{\infty}\Big(4n+|(n+1)(B+1)-(A+1)|\Big)\left(\frac{(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_{n+1}}\right)\\
&= 2(1-\beta)\sum_{n=0}^{\infty}\Big(4(n+1-1)+|(n+1)(B+1)-(A+1)|\Big)\times\\
&\quad\left(\frac{(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_{n+1}}\right)-|B-A|\\
&= 2(1-\beta)\left[4\sum_{n=0}^{\infty}\left(\frac{(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_n}\right)-4\sum_{n=0}^{\infty}\left(\frac{(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_{n+1}}\right)\right.\\
&\quad+|B+1|\sum_{n=0}^{\infty}\left(\frac{(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_n}\right)+|A+1|\sum_{n=0}^{\infty}\left(\frac{(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_{n+1}}\right)\\
&\quad\left.-|B-A|\right]\\
&= 2(1-\beta)\left[4\sum_{n=0}^{\infty}\left(\frac{(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_n}\right)-4\sum_{n=0}^{\infty}\left(\frac{(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_{n+1}}\right)\right.\\
&\quad+|B+1|\sum_{n=0}^{\infty}\left(\frac{(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_n}\right)+|A+1|\sum_{n=0}^{\infty}\left(\frac{(|a|)_n(b)_n(c)_n}{(b+1)_n(c+1)_n(1)_{n+1}}\right)\\
&\quad\left.-|B-A|\right]
\end{aligned}$$

Using the formula (15) and results (4) of Lemma 1.3 in the aforementioned equation, we find that

$$\begin{aligned}
T \leq\ & 2(1-\beta)\left[4\frac{bc\Gamma(1-|a|)}{c-b}\left[\frac{\Gamma(b)}{\Gamma(1-|a|+b)}-\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\right]\right.\\
& -4\left(\frac{bc}{(|a|-1)(b-1)(c-1)}\right)\left[\frac{\Gamma(2-|a|)}{c-b}\left(\frac{(c-1)\Gamma(b)}{\Gamma(1-|a|+b)}-\frac{(b-1)\Gamma(c)}{\Gamma(1-|a|+c)}\right)-1\right]\\
& +|B+1|\frac{bc\Gamma(1-|a|)}{c-b}\left[\frac{\Gamma(b)}{\Gamma(1-|a|+b)}-\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\right]\\
& +|A+1|\left(\frac{bc}{(|a|-1)(b-1)(c-1)}\right)\left[\frac{\Gamma(2-|a|)}{c-b}\left(\frac{(c-1)\Gamma(b)}{\Gamma(1-|a|+b)}-\frac{(b-1)\Gamma(c)}{\Gamma(1-|a|+c)}\right)-1\right]\\
& \left.-|B-A|\right]\\
=\ & 2(1-\beta)\left[4\frac{bc\Gamma(1-|a|)}{c-b}\left[\frac{\Gamma(b)}{\Gamma(1-|a|+b)}-\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\right]\right.\\
& +4\left(\frac{bc}{(b-1)(c-1)}\right)\frac{\Gamma(1-|a|)}{c-b}\left(\frac{(c-1)\Gamma(b)}{\Gamma(1-|a|+b)}-\frac{(b-1)\Gamma(c)}{\Gamma(1-|a|+c)}\right)\\
& +4\left(\frac{bc}{(|a|-1)(b-1)(c-1)}\right)\\
& +|B+1|\frac{bc\Gamma(1-|a|)}{c-b}\left[\frac{\Gamma(b)}{\Gamma(1-|a|+b)}-\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\right]\\
& -|A+1|\left(\frac{bc}{(b-1)(c-1)}\right)\frac{\Gamma(1-|a|)}{c-b}\left(\frac{(c-1)\Gamma(b)}{\Gamma(1-|a|+b)}-\frac{(b-1)\Gamma(c)}{\Gamma(1-|a|+c)}\right)\\
& \left.-|A+1|\left(\frac{bc}{(|a|-1)(b-1)(c-1)}\right)-|B-A|\right]\\
=\ & 2(1-\beta)\left[\frac{bc\Gamma(1-|a|)}{c-b}\left[\left(4+|B+1|+\frac{4-|A+1|}{(b-1)}\right)\frac{\Gamma(b)}{\Gamma(1-|a|+b)}\right.\right.\\
& \left.-\left(4+|B+1|+\frac{4-|A+1|}{(c-1)}\right)\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\right]\\
& \left.(4-|A+1|)\left(\frac{bc}{(|a|-1)(b-1)(c-1)}\right)-|B-A|\right].
\end{aligned}$$

Because of (10), the above expression is bounded above by $|B-A|$ , and hence,

$$\begin{aligned}
& 2(1-\beta)\left[\frac{bc\Gamma(1-|a|)}{c-b}\left[\left(4+|B+1|+\frac{4-|A+1|}{(b-1)}\right)\frac{\Gamma(b)}{\Gamma(1-|a|+b)}\right.\right.\\
& \qquad \left.-\left(4+|B+1|+\frac{4-|A+1|}{(c-1)}\right)\frac{\Gamma(c)}{\Gamma(1-|a|+c)}\right]\\
& \left.+(4-|A+1|)\left(\frac{bc}{(|a|-1)(b-1)(c-1)}\right)-|B-A|\right]\leq|B-A|.
\end{aligned}$$

$\mathscr{I}_{b+1,c+1}^{a,b,c}(f)(z)$ Therefore, the operator $\mathscr{I}_{b+1,c+1}^{a,b,c}(f)(z)$ maps $\mathscr{R}(\beta)$ into $\mathscr{S}_p[A,B]$ and the result follows. □

## Article Information

**Disclaimer (Artificial Intelligence):** The author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc).

**Competing Interests:** Authors have declared that no competing interests exist.

**Funding:** No funding grants to the authors.

# References

[1] P. L. Duren Univalent. *functions (Grundlehren der mathematischen Wis-senschaften 259, New York, Bcerlin, Heidelberg, Tokyo)*. Springer–Verlag, 1983.

[2] A. W. Goodman. Univalent functions, Vol.I and Vol.II. 1983. Tampa Florida iner Publishing Company.

[3] A. W. Goodman. On uniformly convex functions. *Ann. Polon. Math.*, 56(1):87–92, 1991.

[4] A. W. Goodman. On uniformly starlike functions. *J. Math. Anal. Appl.*, 155(2):364–370, 1991.

[5] F. Rønning. Uniformly convex functions and a corresponding class of starlike functions. *Proc. Amer. Math. Soc.*, 118(1):189–196, 1993.

[6] W. C. Ma and D. Minda. Uniformly convex functions. *Ann. Polon. Math.*, 57(2):165–175, 1992.

[7] S. Kanas and W. Wisniowska. Conic regions and k-uniform convexity. *J. Comput. Appl. Math.*, 105:327–336, 1999.

[8] S. Kanas and W. Wisniowska. Conic domains and starlike functions. *Rev. Roumaine Math. Pures Appl.*, 45:647–657, 2000.

[9] W. Janowski. Some extremal problems for certain families of analytic functions I. *Ann. Polon. Math.*, 28(4):297–326, 1973.

[10] Sarfraz Nawaz. *Certain subclasses of analytic functions associated with conic domains*. PhD thesis, Comsats Instttute of Information Technology, Pakistan, 2012.

[11] T. H. MacGregor. Functions whose derivative has a positive real part. *Trans. Amer. Math. Soc.*, 104:532–537, 1962.

[12] S. Ponnusamy and F. Rønning. Geometric properties for convolutions of hyper-geometric functions and functions with the derivative in a halfplane. *Integral Transform. Spec. Funct.*, 8(1-2):121–138, 1999. MR1730602.

[13] S. Ponnusamy and F. Rønning. Starlikeness properties for convolutions involving hypergeometric series. *Ann. Univ. Mariae Curie-Sklodowska Sect. A*, 52(1):141–155, 1998.

[14] Y. C. Kim and S. Ponnusamy. Sufficiency for Gaussian hypergeometric functions to be uniformly convex. *Int. J. Math. Math. Sci.*, 22 (4):765–773, 1999. MR1733277.

[15] R. Kustner. Mapping properties of hypergeometric functions and convolutions of starlike or convex functions of order $\alpha$. *Comput. Methods Funct. Theory*, 2(2):597–610, 2002. [On table of contents: 2004], . MR2038140.

[16] S. S. Miller and P. T. Mocanu. Univalence of Gaussian and confluent hyper-geometric functions. *Proc. Amer. Math. Soc.*, 110(2): 333–342, 1990. MR1017006.

[17] I. R. Nezhmetdinov and S. Ponnusamy. New coefficient conditions for the star-likeness of analytic functions and their applications. *Houston J. Math.*, 31(2):587–604, 2005. MR2132852.

[18] R. Parvatham and D. J. Prabhakaran. On the Hohlov convolution operator of the class $S_p$. *Far East J. Math.Sci*, Special volume: 217–228, 2001.

[19] S. Ponnusamy. Univalence of Alexander transform under new mapping proper-ties, Complex Variables Theory Appl. 30:1, 1996. MR1395231.

[20] S. Ponnusamy. Close-to-convexity properties of Gaussian hypergeometric func-tions. *J. Comput. Appl. Math.*, 88(2):327–337, 1998. MR1613250.

[21] S. Ponnusamy and S. Sabapathy. Geometric properties of generalized hypergeo-metric functions. *Ramanujan J.*, 1(2):187–210, 1997. MR1606192.

[22] S. Ponnusamy and M. Vuorinen. Univalence and convexity properties for Gaus-sian hypergeometric functions. *Rocky Mountain J. Math*, 31(1):327–353, 2001. MR1821384.

[23] H. Silverman. Starlike and convexity properties for hypergeometric functions. *J. Math. Anal. Appl.*, 172(2):574–581, 1993. MR1201006.

[24] T. Sugawa and L.-M. Wang. Geometric properties of the shifted hypergeometric functions. *Complex Anal. Oper. Theory*, 11(8): 1879–1893, 2017. MR3717398.

[25] K. Chandrasekran. *Geometric propeties generalized hypergeometric functions and stable functions*. PhD thesis, Anna University, India, 2021.

[26] A. R. Miller and R. B. Paris. Clausen's series ${}_3F_2(1)$ with integral parameter differences and transformations of the hypergeometric function ${}_2F_2(x)$. *Integral Transforms and Spec*, 23(1):21–33, 2012.

[27] M. A. Shpot and H. M. Srivastava. The Clausenian hypergeometric function ${}_3F_2$ with unit argument and negative integral parameter differences. *Appl. Math. Comput.*, 259:819–827, 2015.