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Research Article

Spatial HyperGraphs and Spatial SuperHyperGraphs

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Abstract

Graph theory studies the mathematical structures of vertices and edges to model relationships and connectivity [1, 2]. Hypergraphs extend this framework by allowing hyperedges to connect arbitrarily many vertices at once [3], and superhypergraphs further generalize hypergraphs via iterated powerset constructions to capture hierarchical linkages among edges [4, 5].

A spatial hypergraph is a hypergraph in which each vertex is assigned a fixed location in Euclidean space through an embedding. In this paper, we introduce the spatial n-SuperHyperGraph, an extension of spatial hypergraphs within the n-SuperHyperGraph framework. This generalization provides a clear and intuitive means of representing the hierarchical structures inherent in spatial graphs, yielding significant advantages for modeling and analysis

1. Introduction

We begin by reviewing the basic terminology and notation used throughout this paper. Unless specified otherwise, all graphs are assumed to be undirected, finite, and simple. For more extensive discussions of particular operations and concepts, the reader is referred to the literature.

1.1. SuperHyperGraph

A *hypergraph* generalizes a standard graph by allowing *hyperedges* that can join any number of vertices simultaneously [6–12]. Extending this idea, a *SuperHyperGraph* incorporates iterated powerset constructions to capture hierarchical relationships among hyperedges, a topic of growing interest in recent studies [13–18]. Practical applications of SuperHyperGraphs include molecular modeling, network analysis, and signal processing [19–23]. In what follows, the integer parameter n in the nth powerset and in an n-SuperHyperGraph always denotes a nonnegative integer.

Definition 1.1 (Base Set)**.** *A* base set *S is the underlying domain from which all further constructions are drawn. Formally,*

$$S = \{x \mid x \text{ belongs to the specified universe}\}.$$

Every element appearing in $\mathscr{P}(S)$ or in iterated powersets $\mathscr{P}_n(S)$ is an element of S.

Definition 1.2 (Powerset)**.** *The* powerset *of a set S, written $\mathscr{P}(S)$, is the collection of all subsets of S, including $\emptyset$ and S itself:*

$$\mathscr{P}(S) = \{A \mid A \subseteq S\}.$$

Example 1.3 (Pizza Topping Combinations)**.** *A pizzeria offers four optional toppings:*

$$S = \{\text{Cheese, Salmon, Mushrooms, Olives}\}.$$

The powerset of S is

$$\mathscr{P}(S) = \big\{\varnothing, \{\text{Cheese}\}, \{\text{Salmon}\}, \{\text{Mushrooms}\},$$

$$\{\text{Olives}\}, \{\text{Cheese}, \text{Salmon}\}, \{\text{Cheese}, \text{Mushrooms}\}, \{\text{Cheese}, \text{Olives}\},$$

$$\{\text{Salmon}, \text{Mushrooms}\}, \{\text{Salmon}, \text{Olives}\}, \{\text{Mushrooms}, \text{Olives}\}, \{\text{Cheese}, \text{Salmon}, \text{Mushrooms}\},$$

$$\{\text{Cheese}, \text{Salmon}, \text{Olives}\}, \{\text{Cheese}, \text{Mushrooms}, \text{Olives}\},$$

$$\{\text{Salmon}, \text{Mushrooms}, \text{Olives}\}, \{\text{Cheese}, \text{Salmon}, \text{Mushrooms}, \text{Olives}\}\big\}.$$

This collection represents every possible combination of toppings a customer can choose.

Definition 1.4 (Hypergraph)**.** *[3, 24] A* hypergraph *$H = (V(H), E(H))$ consists of*

- *A finite vertex set $V(H)$.*
- *A finite collection $E(H)$ of nonempty subsets of $V(H)$, called hyperedges.*

Hypergraphs are well suited to model higher-order interactions among elements of $V(H)$.

Example 1.5 (Unit Test Coverage Hypergraph)**.** *Consider a software project consisting of five functions:*

$$V = \{f_1, f_2, f_3, f_4, f_5\},$$

where each f_i represents a distinct function in the codebase. Suppose we have three unit tests, each of which exercises a specific subset of these functions:

$$T_1 = \{f_1, f_2\}, \quad T_2 = \{f_2, f_3, f_4\}, \quad T_3 = \{f_1, f_3, f_5\}.$$

Defining the hyperedge set by

$$E = \{T_1, T_2, T_3\},$$

we obtain the hypergraph

$$H = \big(V, E\big).$$

Here each hyperedge T_i corresponds to a unit test covering exactly the functions in T_i, so H models the higher-order relationship between test cases and the functions they exercise.

Definition 1.6 (n-th Powerset)**.** *[25–29] The n-th powerset of a set X, denoted $P_n(X)$, is defined by:*

$$P_1(X) = \mathscr{P}(X), \qquad P_{n+1}(X) = \mathscr{P}\big(P_n(X)\big), \quad n \geq 1.$$

The corresponding nonempty powerset $P_n^*(X)$ is obtained by iterating $\mathscr{P}^*(\cdot)$, where $\mathscr{P}^*(Y) = \mathscr{P}(Y) \setminus \{\emptyset\}$.

Example 1.7 (Hierarchical Feature Subset Selection)**.** *In a machine-learning pipeline, let the base feature set be*

$$X = \{Age,\ Income,\ Education\}.$$

The first-level powerset

$$P_1(X) = \mathscr{P}(X) = \big\{\varnothing, \{Age\}, \{Income\}, \{Education\},$$

$$\{Age{,}Income\}, \{Age{,}Education\}, \{Income{,}Education\},$$

$$\{Age{,}Income{,}Education\}\big\}$$

represents all possible feature subsets that can be used to train base models.
At the second level, we form ensembles of feature-subset models by taking subsets of $P_1(X)$. For example, consider

$$E_1 = \big\{\{Age\}, \{Income{,}Education\}\big\}$$

$$\text{and} \quad E_2 = \big\{\{Age{,}Education\}, \{Income\}\big\},$$

each of which is an element of

$$P_2(X) = \mathscr{P}\big(P_1(X)\big).$$

Here E_1 corresponds to an ensemble combining a model trained on $\{Age\}$ with another trained on $\{Income{,}Education\}$, and similarly for E_2.
More generally, the n-th powerset $P_n(X)$ captures nested hierarchies of model combinations:

$$P_{n+1}(X) = \mathscr{P}\big(P_n(X)\big), \quad n \geq 1,$$

enabling multilevel ensemble architectures in which ensembles themselves are grouped and re-combined at higher levels.

Definition 1.8 (n-SuperHyperGraph)**.** *[22, 30, 31] Let V_0 be a finite base set. Define iteratively*

$$\mathscr{P}^0(V_0) = V_0, \quad \mathscr{P}^{k+1}(V_0) = \mathscr{P}\big(\mathscr{P}^k(V_0)\big).$$

*An n-*SuperHyperGraph *is a pair*

$$\mathrm{SuHG}^{(n)} = (V, E), \quad V, E \subseteq \mathscr{P}^n(V_0),$$

where each element of V is called an n-supervertex and each element of E an n-superedge.

Example 1.9 (Manufacturing Assembly Hierarchy)**.** *Consider a simple assembly plant in which individual machines are organized into work cells and those cells form production lines. Let the base set of machines be*

$$V_0 = \{M_1, M_2, M_3, M_4\},$$

where each M_i denotes a distinct machine. We form first-level groups (work cells) by

$$C_1 = \{M_1, M_2\}, \quad C_2 = \{M_3, M_4\},$$

so that $C_1, C_2 \in \mathscr{P}^1(V_0)$. Next, we aggregate these cells into a second-level group (production line):

$$L = \{C_1, C_2\} \in \mathscr{P}^2(V_0).$$

Set

$$V = \{L\}, \quad E = \{L\}.$$

Then

$$\mathrm{SuHG}^{(2)} = (V_0, V, E)$$

is a 2-SuperHyperGraph modeling the two-tier hierarchy of the plant: machines $\to$ cells $\to$ line. Here L serves as both a 2-supervertex and a 2-superedge, illustrating how individual machines are nested into cells, which are in turn nested into a production line.

Example 1.10 (Hierarchical Ontology in an AI Knowledge Graph)**.** *Consider a simple ontology used in an AI knowledge graph. Let the base set of atomic concepts be*

$$V_0 = \{\text{dog, cat, car, truck}\}.$$

At the first level, we form concept clusters (synsets):

$$C_1 = \{\text{dog}, \text{cat}\}, \quad C_2 = \{\text{car}, \text{truck}\} \in \mathscr{P}^1(V_0).$$

At the second level, we aggregate these clusters into a domain:

$$D = \{C_1, C_2\} \in \mathscr{P}^2(V_0).$$

Finally, set

$$V = \{D\}, \quad E = \{D\}.$$

Then

$$\mathrm{SuHG}^{(2)} = \big(V_0, V, E\big)$$

is a 2-SuperHyperGraph that models the two-tier hierarchy in the ontology: atomic concepts $\to$ synsets $\to$ domain.

1.2. Spatial hypergraphs

A Spatial Hypergraph is a hypergraph where each vertex is assigned a fixed location in Euclidean space via an embedding (cf.[32–36]).

Definition 1.11 (Spatial Hypergraph)**.** *Let $d \in \mathbb{N}$. A d-*dimensional spatial hypergraph *is a triple*

$$\mathscr{H} = \big(V, S, \lambda\big),$$

consisting of

(a) a finite vertex set V*;*

(b) a finite family of non-empty subsets $S \subseteq \mathscr{P}(V) \setminus \{\varnothing\}$, whose members are called hyperedges*; and*

(c) an embedding map $\lambda : V \longrightarrow \mathbb{R}^d$ *that assigns each vertex a fixed position in Euclidean space.*

We call $\lambda(v)$ the location *of vertex v. Unless stated otherwise, λ is assumed to be injective (no two vertices occupy the same point). The pair (V, S) is the underlying hypergraph; the map λ endows it with spatial geometry.*

Remark 1.12. *The case $d = 2$ yields a* planar spatial hypergraph*, often simply called a* spatial hypergraph *in the literature when all vertices lie in the plane and have pre-assigned coordinates Higher-dimensional variants ($d > 2$) arise naturally in applications such as volume data analysis or molecular modelling.*

Example 1.13 (Urban Bus Transit Network)**.** *Consider a small city's bus network embedded in the plane ($d = 2$). Let the set of bus stops be*

$$V = \{s_1, s_2, s_3, s_4, s_5\},$$

with embedding

$$\lambda(s_1) = (0,0), \quad \lambda(s_2) = (2,1), \quad \lambda(s_3) = (4,0),$$
$$\lambda(s_4) = (3,3), \quad \lambda(s_5) = (1,3) \in \mathbb{R}^2.$$

Define the family of hyperedges (bus routes) by

$$S = \{\{s_1, s_2, s_3\}, \{s_2, s_4, s_5\}, \{s_1, s_5\}\}.$$

Then

$$\mathscr{H} = (V, S, \lambda)$$

is a 2-dimensional spatial hypergraph. Here each hyperedge represents the set of stops served by a particular bus line, and λ specifies the geographic location of each stop.

2. Main Results: Spatial SuperHyperGraphs

As a principal result of this paper, we introduce the notion of *Spatial SuperHyperGraphs*, which extends the concept of spatial graphs within the framework of SuperHyperGraphs.

Definition 2.1 (Spatial n-SuperHyperGraph)**.** *Let $d, n \in \mathbb{N}$ and let V_0 be a finite base set. A* d-dimensional spatial n-SuperHyperGraph *is a quadruple*

$$\mathscr{G}^{(n)} = (V_0, V, E, \lambda),$$

where

(a) (V, E) is an n-SuperHyperGraph over V_0, i.e. $V, E \subseteq \mathscr{P}^n(V_0)$;
(b) $\lambda : V_0 \to \mathbb{R}^d$ is an injective embedding map assigning each base element a fixed position in Euclidean space.

Example 2.2 (Water Distribution Control Zones)**.** *Consider a water distribution network laid out on a map ($d = 2$). Let the base set of control valves be*

$$V_0 = \{v_1, v_2, v_3, v_4, v_5\},$$

with geographic coordinates

$$\lambda(v_1) = (1,1), \quad \lambda(v_2) = (1,4), \quad \lambda(v_3) = (4,1),$$
$$\lambda(v_4) = (4,4), \quad \lambda(v_5) = (2.5,2.5) \in \mathbb{R}^2.$$

We first group valves into local zones *(the 1st powerset):*

$$Z_1 = \{v_1, v_2, v_5\}, \quad Z_2 = \{v_3, v_4, v_5\}.$$

Next, we aggregate zones into a higher-level sector *(the 2nd powerset):*

$$S = \{Z_1, Z_2\}.$$

Set

$$V = \{S\}, \quad E = \{S\}.$$

Then

$$\mathscr{G}^{(2)} = (V_0, V, E, \lambda)$$

is a 2-dimensional spatial 2-SuperHyperGraph. In this model, each valve has a fixed location; valves combine to form overlapping zones, and those zones jointly define a single sector, illustrating a two-level spatial hierarchy in the control network.

Example 2.3 (Smart Building Sensor Hierarchy)**.** *Consider a simple smart-building deployment in the plane ($d = 2$). Let the base set of sensors be*

$$V_0 = \{s_1, s_2, s_3, s_4\},$$

equipped with the embedding

$$\lambda(s_1) = (0,0), \quad \lambda(s_2) = (0,4), \quad \lambda(s_3) = (5,0), \quad \lambda(s_4) = (5,4) \in \mathbb{R}^2.$$

We organize the sensors into rooms (the 1st powerset):

$$R_1 = \{s_1, s_2\}, \quad R_2 = \{s_3, s_4\},$$

and then into floors (the 2*nd powerset):*

$$F_1 = \{R_1\}, \quad F_2 = \{R_2\}.$$

Define

$$V = \{F_1, F_2\} \quad \text{and} \quad E = \{F_1, F_2\}.$$

Then

$$\mathscr{G}^{(2)} = (V_0, V, E, \lambda)$$

is a 2*-dimensional spatial* 2*-SuperHyperGraph. Here each floor F_i plays the role of both a supervertex and a superedge, illustrating how rooms and floors form a two-level hierarchy over the sensor network.*

Theorem 2.4. *Let $\mathscr{G}^{(n)} = (V_0, V, E, \lambda)$ be a d-dimensional spatial n-SuperHyperGraph. Then the underlying pair (V, E) is an n-SuperHyperGraph over V_0.*

Proof. By Definition 2.1, we have $V, E \subseteq \mathscr{P}^n(V_0)$. These are exactly the conditions required for (V, E) to be an n-SuperHyperGraph on the base set V_0. □

Theorem 2.5. *The class of d-dimensional spatial hypergraphs (Definition 1.11) embeds into the class of d-dimensional spatial* 1*-SuperHyperGraphs. Concretely, every spatial hypergraph $\mathscr{H} = (V_0, S, \lambda)$ gives rise to a spatial* 1*-SuperHyperGraph $\mathscr{G}^{(1)} = (V_0, V, E, \lambda)$ by*

$$V = \big\{\{v\} \mid v \in V_0\big\}, \quad E = S.$$

Conversely, any spatial 1*-SuperHyperGraph with $V = \{\{v\} : v \in V_0\}$ corresponds uniquely to a spatial hypergraph.*

Proof. Suppose $\mathscr{H} = (V_0, S, \lambda)$ is a spatial hypergraph. Define

$$V = \{\{v\} \subseteq V_0 : v \in V_0\}$$

$$\text{and} \quad E = S \subseteq \mathscr{P}(V_0).$$

Then $V, E \subseteq \mathscr{P}^1(V_0)$, so (V, E) is a 1-SuperHyperGraph over V_0. Together with λ, this yields a spatial 1-SuperHyperGraph as in Definition 2.1.
Conversely, let $\mathscr{G}^{(1)} = (V_0, V, E, \lambda)$ be a spatial 1-SuperHyperGraph with $V = \{\{v\} : v \in V_0\}$. Identifying each singleton $\{v\}$ with the base vertex v recovers a spatial hypergraph (V_0, E, λ). This correspondence is clearly inverse to the construction above, proving the claim. □

Theorem 2.6 (Hierarchy Projection)**.** *Let $\mathscr{G}^{(n)} = (V_0, V, E, \lambda)$ be a d-dimensional spatial n-SuperHyperGraph. For each k with $1 \le k \le n$, define*

$$V^{(k)} = V \cap \mathscr{P}^k(V_0), \quad E^{(k)} = E \cap \mathscr{P}^k(V_0),$$

$$\lambda^{(k)} = \lambda\Big|_{\bigcup V^{(k)}}.$$

Then the quadruple

$$\mathscr{G}^{(k)} = \big(V_0,\, V^{(k)},\, E^{(k)},\, \lambda^{(k)}\big)$$

is a d-dimensional spatial k-SuperHyperGraph.

Proof. We verify the three parts of Definition 2.1 for level k.
(a) Subsets of the k-th powerset. By construction,

$$V^{(k)} = V \cap \mathscr{P}^k(V_0)$$

$$\text{and} \quad E^{(k)} = E \cap \mathscr{P}^k(V_0).$$

Since $V, E \subseteq \mathscr{P}^n(V_0)$ and $k \le n$, we have $\mathscr{P}^k(V_0) \subseteq \mathscr{P}^n(V_0)$. Hence

$$V^{(k)}, E^{(k)} \subseteq \mathscr{P}^k(V_0),$$

so $(V^{(k)}, E^{(k)})$ is by definition a k-SuperHyperGraph over V_0.
(b) Embedding restriction. The original embedding $\lambda : V_0 \to \mathbb{R}^d$ is injective. Its restriction $\lambda^{(k)} = \lambda|_{\bigcup V^{(k)}}$ thus remains injective on the (possibly smaller) domain $\bigcup V^{(k)} \subseteq V_0$.
(c) Structure of the quadruple. Combining (a) and (b), we see that $\big(V_0,\, V^{(k)},\, E^{(k)},\, \lambda^{(k)}\big)$ satisfies all requirements of Definition 2.1 with n replaced by k. Therefore $\mathscr{G}^{(k)}$ is a d-dimensional spatial k-SuperHyperGraph. □

Theorem 2.7 (Induced Substructure)**.** *Let $\mathscr{G}^{(n)} = (V_0, V, E, \lambda)$ be a d-dimensional spatial n-SuperHyperGraph, and let $U \subseteq V_0$ be nonempty. Define*

$$V|_U = \{X \in V : X \subseteq \mathscr{P}^n(U)\},$$
$$E|_U = \{e \in E : e \subseteq \mathscr{P}^n(U)\}, \quad \lambda|_U = \lambda\big|_U.$$

Then

$$\mathscr{G}^{(n)}[U] = \big(U, V|_U, E|_U, \lambda|_U\big)$$

is a d-dimensional spatial n-SuperHyperGraph on the base set U.

Proof. Since $U \subseteq V_0$, by iterating the monotonicity of the powerset operator we have

$$\mathscr{P}^n(U) \subseteq \mathscr{P}^n(V_0).$$

Therefore every $X \in V|_U$ and every $e \in E|_U$ is also an element of $\mathscr{P}^n(V_0)$, so

$$V|_U, E|_U \subseteq \mathscr{P}^n(U),$$

and in particular $V|_U, E|_U \subseteq \mathscr{P}^n(V_0)$. Since (V, E) was an n-SuperHyperGraph over V_0, it follows that $(V|_U, E|_U)$ satisfies the same axioms over the smaller base set U.
Next, $\lambda|_U : U \to \mathbb{R}^d$ remains injective because λ was injective on V_0 and $U \subseteq V_0$. Thus $(U, V|_U, E|_U, \lambda|_U)$ meets all the requirements of Definition 2.1, confirming that $\mathscr{G}^{(n)}[U]$ is indeed a spatial n-SuperHyperGraph on U. □

Theorem 2.8 (Underlying Spatial Hypergraph)**.** *Let $\mathscr{G}^{(n)} = (V_0, V, E, \lambda)$ be a spatial n-SuperHyperGraph, and let*

$$V^{(1)} = V \cap \mathscr{P}^1(V_0),$$
$$E^{(1)} = E \cap \mathscr{P}^1(V_0), \quad \lambda^{(1)} = \lambda\big|_{V_0}.$$

If $V^{(1)} = \{\{v\} : v \in V_0\}$, then the triple $(V_0, E^{(1)}, \lambda^{(1)})$ is a d-dimensional spatial hypergraph (Definition 1.11).

Proof. By Theorem 2.6 with $k = 1$, $(V^{(1)}, E^{(1)})$ is a 1-SuperHyperGraph and hence $V^{(1)}, E^{(1)} \subseteq \mathscr{P}(V_0)$. Under the additional assumption $V^{(1)} = \{\{v\} : v \in V_0\}$, each singleton $\{v\}$ identifies uniquely with the base vertex v. Removing the singleton brackets yields

$$V_0 = \{v : \{v\} \in V^{(1)}\},$$
$$E^{(1)} \subseteq \mathscr{P}(V_0).$$

Moreover, $\lambda^{(1)}$ is injective on V_0 since it coincides with λ. Therefore $(V_0, E^{(1)}, \lambda^{(1)})$ satisfies all conditions of Definition 1.11, establishing it as the underlying spatial hypergraph. □

Definition 2.9 (Recursive Convex Region)**.** *For each $k \geq 0$ define a map R_k assigning to every $X \in \mathscr{P}^k(V_0)$ a convex region in $\mathbb{R}^d$ by:*

$$R_0(v) = \{\lambda(v)\}, \quad v \in V_0,$$
$$R_k(X) = \operatorname{Conv}\Big(\bigcup_{Y \in X} R_{k-1}(Y)\Big).$$

Theorem 2.10 (Convex-Hull Recursion)**.** *For each $1 \leq k \leq n$ and each k-superedge $e \in E^{(k)}$, the region $R_k(e)$ defined above equals*

$$R_k(e) = \operatorname{Conv}\Big(\bigcup_{v \in \bigcup_{X \in e} \cdots \bigcup_{Z \in X} Z} \{\lambda(v)\}\Big),$$

where the nested unions descend k levels from e down to base vertices in V_0.

Proof. We proceed by induction on k.
Base case ($k = 1$). If $e \in E^{(1)} \subseteq \mathscr{P}(V_0)$, then

$$R_1(e) = \operatorname{Conv}\big(\bigcup_{v \in e} R_0(v)\big)$$
$$= \operatorname{Conv}\big(\{\lambda(v) : v \in e\}\big),$$

which matches the stated formula.
Inductive step. Suppose the formula holds for level $k-1$. Let $e \in E^{(k)} \subseteq \mathscr{P}^k(V_0)$. By definition,

$$R_k(e) = \operatorname{Conv}\Big(\bigcup_{X \in e} R_{k-1}(X)\Big).$$

By the inductive hypothesis each $R_{k-1}(X)$ is the convex hull of the union of base-point images of those vertices reachable by descending $k-1$ levels from X. Therefore

$$R_k(e) = \operatorname{Conv}\Big(\bigcup_{X \in e} \operatorname{Conv}\big(\{\lambda(v) : v \in \operatorname{Base}(X)\}\big)\Big)$$
$$= \operatorname{Conv}\Big(\{\lambda(v) : v \in \operatorname{Base}(e)\}\Big),$$

where $\operatorname{Base}(e) = \bigcup_{X \in e} \operatorname{Base}(X)$ collects all base vertices under e. This completes the induction. □

Definition 2.11 (Diameter of a Superedge). *For a k-superedge $e \in E^{(k)}$, define its* diameter *by*

$$\operatorname{diam}(e) = \max_{u,v \in \operatorname{Base}(e)} \left\| \lambda(u) - \lambda(v) \right\|,$$

where $\operatorname{Base}(e) \subseteq V_0$ *is as in Theorem 2.10.*

Theorem 2.12 (Diameter Monotonicity). *For each $1 \le k < n$, every $(k+1)$-superedge $e \in E^{(k+1)}$, and every $X \in e$, we have*

$$\operatorname{diam}(X) \le \operatorname{diam}(e).$$

Hence diameters of superedges do not decrease as one ascends the hierarchy.

Proof. Fix $e \in E^{(k+1)}$ and $X \in e \subseteq \mathscr{P}^k(V_0)$. By definition, $\operatorname{Base}(X) \subseteq \operatorname{Base}(e)$. Therefore

$$\operatorname{diam}(X) = \max_{u,v \in \operatorname{Base}(X)} \|\lambda(u) - \lambda(v)\|$$
$$\le \max_{u,v \in \operatorname{Base}(e)} \|\lambda(u) - \lambda(v)\| = \operatorname{diam}(e),$$

as claimed. □

Theorem 2.13 (Affine Invariance). *Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an affine isomorphism (i.e. $T(x) = Ax + b$ with A invertible). Then*

$$\mathscr{G}'^{(n)} = \big(V_0, V, E, T \circ \lambda\big)$$

is also a d-dimensional spatial n-SuperHyperGraph, and $\mathscr{G}'^{(n)}$ is isomorphic to $\mathscr{G}^{(n)}$.

Proof. Since T is bijective and λ is injective, the composition $T \circ \lambda : V_0 \to \mathbb{R}^d$ remains injective. The combinatorial data (V,E) is unchanged, so $(V,E) \subseteq \mathscr{P}^n(V_0)$ still holds. Thus $\mathscr{G}'^{(n)}$ satisfies Definition 2.1.
To see the isomorphism, note that the identity map on V_0 induces a bijection of supervertices and superedges, and T carries Euclidean relations faithfully. Hence $\mathscr{G}'^{(n)}$ and $\mathscr{G}^{(n)}$ are isomorphic as spatial n-SuperHyperGraphs. □

Theorem 2.14 (Powerset Associativity). *For any set V_0 and integers $m,n \ge 0$,*

$$\mathscr{P}^{n+m}(V_0) = \mathscr{P}^m\big(\mathscr{P}^n(V_0)\big).$$

Consequently, if (V_n,E_n) is an n-SuperHyperGraph over V_0 and (V'_m,E'_m) is an m-SuperHyperGraph over the base set V_n, then (V'_m,E'_m) defines an $(n+m)$-SuperHyperGraph over V_0. The same holds in the spatial setting.

Proof. We prove the equality of iterated powersets by induction on m:
Base case $(m = 0)$.

$$\mathscr{P}^{n+0}(V_0) = \mathscr{P}^n(V_0)$$

and

$$\mathscr{P}^0(\mathscr{P}^n(V_0)) = \mathscr{P}^n(V_0)$$

by definition.
Inductive step. Assume $\mathscr{P}^{n+m}(V_0) = \mathscr{P}^m(\mathscr{P}^n(V_0))$. Then

$$\mathscr{P}^{n+(m+1)}(V_0) = \mathscr{P}\big(\mathscr{P}^{n+m}(V_0)\big)$$
$$= \mathscr{P}\Big(\mathscr{P}^m\big(\mathscr{P}^n(V_0)\big)\Big)$$
$$= \mathscr{P}^{m+1}\big(\mathscr{P}^n(V_0)\big),$$

completing the induction.
For the corollary, observe that

$$(V'_m, E'_m) \subseteq \mathscr{P}^m(V_n)$$
$$= \mathscr{P}^m\big(\mathscr{P}^n(V_0)\big)$$
$$= \mathscr{P}^{n+m}(V_0),$$

so (V'_m,E'_m) is an $(n+m)$-SuperHyperGraph over V_0. If in addition $\lambda : V_0 \to \mathbb{R}^d$ is an embedding for level n, it remains so at level $n+m$, yielding a spatial $(n+m)$-SuperHyperGraph. □

Theorem 2.15 (Non-Empty Intersection Closure). *Let $\mathscr{G}^{(n)} = (V_0,V,E,\lambda)$ be a spatial n-SuperHyperGraph. Suppose $e_1,e_2 \in E$ are two n-superedges with $e_1 \cap e_2 \neq \varnothing$. Then*

$$e_1 \cap e_2 \in \mathscr{P}^n(V_0)$$
$$\text{and} \quad e_1 \cap e_2 \subseteq \bigcup_{i=1,2} \operatorname{Base}(e_i),$$

so $e_1 \cap e_2$ can be viewed as a valid n-superedge in the underlying powerset. In particular, the family of non-empty intersections of superedges forms a sub-SuperHyperGraph.

Proof. Since $e_1,e_2 \subseteq \mathscr{P}^n(V_0)$, their set-theoretic intersection $e_1 \cap e_2$ also lies in $\mathscr{P}^n(V_0)$. Non-emptiness follows from the hypothesis. Moreover, every element of $e_1 \cap e_2$ is itself a $(n-1)$-supervertex (or lower) nested within both e_1 and e_2, hence lies in $\bigcup_{i=1,2} \operatorname{Base}(e_i)$. Thus the intersections inherit the superedge property and form an n-SuperHyperGraph on V_0. The embedding λ is untouched, so this remains valid in the spatial context. □

3. Conclusion and Future Works

In this paper, we introduced the concept of the spatial n-SuperHyperGraph, which generalizes spatial hypergraphs within the framework of n-SuperHyperGraphs. We hope that future research will explore further extensions by incorporating structures such as fuzzy sets [37–39], rough Set[40–42], soft sets [43, 44], hyperrough sets [14, 45, 46], hyperfuzzy sets [47–49], hesitant fuzzy sets [50, 51], and neutrosophic sets [52–54]. We also look forward to further advances in algorithm design and computational experiments for spatial n-SuperHyperGraphs.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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