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Research Article

Advancing Numerical Evaluation of Cauchy Principal Value Integrals Using a Mixed Quadrature Rule

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Abstract:

In this Paper, a new derivative-based quadrature rule designed for evaluating complex Cauchy principal value integrals that posses' singularities within the integration domain. A new quadrature rule has been constructed by modifying Simpson's rule on the basis of Cauchy principal value. Further We utilize Cubic Hermite Interpolation to construct a quadrature rule. Both rules are of precision degree 4, which is reflected in the error analysis. Furthermore by utilizing the error associated with the two rules, a combined rule of degree of precision 6 has been developed. This new rule yields lower errors compared to the other rules while maintaining minimal runtime, as verified using several test integrals.

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1. Introduction

In calculus and complex analysis, evaluating integrals that contain singularities within the domain of integration presents a significant analytical difficulty. A singularity occurs at a point where the integrand becomes undefined or unbounded, often rendering the integral divergent in the standard sense. Since the integrand in principal value integrals becomes singular at specific points, the Cauchy principal value problem is regarded as a key challenge within the broader class of singular integral problems. The main objective of this paper is to evaluate the Cauchy principal value integral [1, 2] of the form

$$\int_{-1}^{1} \frac{f(z)}{z} dz \tag{1.1}$$

where $f(z)$ is analytic function in a simply connected domain.

In 1979, Acharya and Das [3] proposed a modified rule derived from a pair of formulas originally introduced by Price [4]. In 1984, Milovanovic et al [5] worked on evaluating Cauchy principal value integrals using interpolatory rules. In 2023, S. Mahesar et al [6] investigated on Centroidal mean derivative based quadrature rules. In 2017, Legua and Sánchez-Ruiz [7] proposed a generalized definition of the Cauchy principal value for complex contour integrals, extending earlier residue-based approaches originally used in thermodynamic entropy calculations. In 2025, B. N. Nayak et al. [8] introduced an improved hybrid quadrature rule that significantly enhances the numerical accuracy for evaluating singular complex-valued integrals. Subsequent researchers [9, 10, 11, 12, 13] also developed effective schemes for computing Cauchy principal value (CPV) integrals.

Within the existing literature, approaches such as Richardson extrapolation and Kronrod extensions [1] are known to enhance the accuracy of numerical rules, though they often involve considerable complexity. To address this, R. N. Das and G. Pradhan [14] proposed a simpler mixed-quadrature strategy in 1996. Building on this idea, the authors in [15, 16, 17] have likewise improved accuracy by blending elementary quadrature rules. In 2022, S. K. Mohanty and R. B. Dash [18] further generalized the mixed-quadrature framework. Inspired by these earlier works, a new derivative-based quadrature rule has been constructed by modifying Simpson's rule on the basis of Cauchy principal value. Further by utilizing Cubic Hermite Interpolation, a new rule has been constructed. Both rules are of precision degree 4, which is reflected in the error analysis. Furthermore by utilizing the error associated with the two rules, a combined rule of higher degree has been developed. This new rule yields lower errors compared to the other rules while maintaining minimal runtime, which is verified by using several test integrals.

2. Formation of Modified Simpson's 1/3 on the Basis of Cauchy Principal Value

In this section a new derivative-based quadrature rule has been constructed by modifying Simpson's rule [1] on the basis of Cauchy principal value. Consider the CPV integral

$$\mathrm{I} = \int_{-1}^{1} \frac{f(z)}{z}\, dz = \lim_{\epsilon\to 0}\left[\int_{-1}^{0-\epsilon} \frac{f(z)}{z}\, dz + \int_{0+\epsilon}^{1} \frac{f(z)}{z}\, dz\right] \tag{2.1}$$

Applying Simpson's $1/3$ Rule to each of the integral present on the right hand side of the above equation, we have

$$RS_{1/3}(f) = \int_{-1}^{1} \frac{f(z)}{z}\, dz = \lim_{\epsilon\to 0}\left[\frac{1-\epsilon}{6}\left\{\frac{f(-1)}{-1} + 4\,\frac{f(-\frac{(1+\epsilon)}{2})}{\frac{-(1+\epsilon)}{2}} + \frac{f(-\epsilon)}{-\epsilon}\right\}\right]$$
$$+\lim_{\epsilon\to 0}\left[\frac{1-\epsilon}{6}\left\{\frac{f(1)}{1} + 4\,\frac{f(\frac{(1+\epsilon)}{2})}{\frac{(1+\epsilon)}{2}} + \frac{f(\epsilon)}{\epsilon}\right\}\right]$$

$$\Rightarrow RS_{1/3}(f) = \frac{1}{6}\left[f(1) - f(-1) + \ 2f'(0) + 8\left\{f\left(\frac{1}{2}\right) - f\left(\frac{-1}{2}\right)\right\}\right] \tag{2.2}$$

3. Formation of Hermite Interpolation Based Rule

Let f(z) be a function which is approximated by Hermite interpolation polynomial of degree 3 at the points z_1 = -1,z_2 = 1. That is f(z_i)=$H_3(z_i)$, for i=1,2.

$$\mathrm{H}_3(\mathrm{z}) = \sum_{i=1}^{n} f(z_i)h_i(z) + \sum_{i=1}^{n} f'(z_i)\overline{h_i(z)}$$

$$\mathrm{H}_3(\mathrm{z}) = f(-1)h_1(z) + f(1)h_2(z) + f'(-1)\overline{h_1(z)} + f'(1)\overline{h_2(z)} \tag{3.1}$$

Now

$$\mathrm{l}_1(\mathrm{z}) = \frac{z-z_2}{z_1-z_2} = \frac{z-1}{-1-1} = -\frac{1}{2}(z-1)$$

$$\mathrm{l}_2(\mathrm{z}) = \frac{z-z_1}{z_2-z_1} = \frac{z-(-1)}{1-(-1)} = \frac{1}{2}(z+1)$$

$$\mathrm{l'}_1(\mathrm{z}) = \frac{-1}{2},\quad \mathrm{l'}_2(\mathrm{z}) = \frac{1}{2}$$

$$\mathrm{h}_1(\mathrm{z}) = [1 - 2l'_1(z_1)(z-z_1)](l_1(z))^2]$$

$$= [1 - 2(\frac{-1}{2})(z-(-1))](\frac{(z-1)^2}{4})]$$

$$= (z+2)\frac{(z-1)^2}{4}$$

$$h_2(z) = [1 - 2l'_2(z_2)(z - z_2)](l_2(z))^2]$$

$$= [1 - 2(\tfrac{1}{2})(z-1)](\tfrac{(z+1)^2}{4})]$$

$$= [(2 - z)(\tfrac{(z+1)^2}{4})]$$

$$\overline{h_1(z)} = (z - z_1)](l_1\ (z))^2 = \frac{(z-1)^2(z+1)}{4}$$

$$\overline{h_2(z)} = (z - z_2)](l_2(z))^2 = \frac{(z+1)^2(z-1)}{4}$$

From equation (3.1) We have

$$\frac{H_3(z)}{z} = f(-1)\frac{h_1(z)}{z} + f(1)\frac{h_2(z)}{z} + f'(-1)\ \frac{\overline{h_1(z)}}{z} + f'(1)\frac{\overline{h_2(z)}}{z}$$

Integrating above equation from -1 to 1 we have

$$\int_{-1}^{1} \frac{H_3(z)}{z}\ dz = \int_{-1}^{1} \left[f(-1)\frac{h_1(z)}{z} + f(1)\frac{h_2(z)}{z} + f'(-1)\frac{\overline{h_1(z)}}{z} + f'(1)\frac{\overline{h_2(z)}}{z}\right] dz$$

$$\Rightarrow \int_{-1}^{1} \frac{H_3(z)}{z}\ dz = \frac{4}{3}[f(1) - f(-1)] - \frac{1}{3}[f'(-1) + f'(1)]$$

$$\Rightarrow \int_{-1}^{1} \frac{f(z)}{z}\ dz \approx \int_{-1}^{1} \frac{H_3(z)}{z}\ dz$$
$$\approx \frac{4}{3}[f(1) - f(-1)] - \frac{1}{3}[f'(-1) + f'(1)] = R_3(f)$$

$$So,\quad R_3(f) \approx \frac{4}{3}[f(1) - f(-1)] - \frac{1}{3}[f'(-1) + f'(1)] \tag{3.2}$$

4. Formation of Higher Precision Mixed Rule

Expandig f(1) and f(-1) about '0' using taylor's series at z=0

$$f(1) = f(0) + f'(0) + \frac{1}{2!}f''(0) + \frac{1}{3!}f'''(0) + \frac{1}{4!}f^{iv}(0) + \frac{1}{5!}f^{v}(0) + \ldots$$

$$f(-1) = f(0) - f'(0) + \frac{1}{2!}f''(0) - \frac{1}{3!}f'''(0) + \frac{1}{4!}f^{iv}(0) - \frac{1}{5!}f^{v}(0) + \ldots$$

Now

$$R_1(f) = f(1) - f(-1) = 2f'(0) + \frac{1}{3!}f'''(0) + \frac{1}{5!}f^{v}(0) + \frac{1}{7!}f^{vii}(0) + \ldots$$

Integrating both sides of the above equation from -1 to 1 with respect to z ,

We Have

$$\int_{-1}^{1} \frac{f(z)}{z}\ dz = \int_{-1}^{1} [\frac{f(0)}{z} + \frac{zf'(0)}{z} + \frac{z^2 f''(0)}{2!z} + \frac{z^3 f'''(0)}{3!z} + \frac{z^4 f^{iv}(0)}{4!z} + \ldots\ldots]\ dz$$

$$\int_{-1}^{1} \frac{f(z)}{z}\ dz = 2f'(0) + \frac{1}{9}f'''(0) + \frac{2}{5.5!}f^{v}(0) + \frac{2}{7.7!}f^{vii}(0) + \ldots$$

Expanding $f\left(\frac{1}{2}\right)$ and $f\left(\frac{-1}{2}\right)$ about '0' Using Taylors Series at x=0

$$f(\tfrac{1}{2}) = f(0) + (\tfrac{1}{2})f'(0) + (\tfrac{1}{2})^2\frac{f''(0)}{2!} + (\tfrac{1}{2})^3\frac{f'''(0)}{3!} + (\tfrac{1}{2})^4\frac{f^{iv}(0)}{4!} + \ldots$$

$$f(\tfrac{-1}{2}) = f(0) - (\tfrac{1}{2})f'(0) + (\tfrac{1}{2})^2\frac{f''(0)}{2!} - (\tfrac{1}{2})^3\frac{f'''(0)}{3!} + (\tfrac{1}{2})^4\frac{f^{iv}(0)}{4!} - \ldots\ldots$$

Now We get

$$RS_{1/3}(f) = \frac{1}{6}[f(1) - f(-1) + 2f'(0) + 8\{f(\tfrac{1}{2}) - f(\tfrac{-1}{2})\}]$$

$$= \frac{1}{6}[12f'(0) + 4\frac{f'''(0)}{3!} + \frac{5}{2}\frac{f^{v}(0)}{5!} + + \frac{17}{8}\frac{f^{vii}(0)}{7!} + \frac{65}{32}\frac{f^{ix}(0)}{9!} + \ldots\ldots]$$

$$= 2f'(0) + \frac{2}{3}\frac{f'''(0)}{3!} + \frac{5}{2.6}\frac{f^{v}(0)}{5!} + + \frac{17}{8.6}\frac{f^{vii}(0)}{7!} + \frac{65}{32.6}\frac{f^{ix}(0)}{9!} + \ldots$$

Hence the error associated with the rule $RS_{1/3}(f)$, which is denoted as $ERS_{1/3}(f)$, and is given by

$$ERS_{1/3}(f) = \int_{-1}^{1}\frac{f(z)}{z}\,dz - RS_{1/3}(f)$$

$$= (\frac{2}{5} - \frac{5}{2.6})\frac{f^{v}(0)}{5!} + (\frac{2}{7} - \frac{17}{8.6})\frac{f^{vii}(0)}{7!} + (\frac{2}{9} - \frac{65}{32.6})\frac{f^{ix}(0)}{9!} + \ldots$$

$$ERS_{1/3}(f) = \frac{-1}{60}\frac{f^{v}(0)}{5!} + (\frac{2}{7} - \frac{17}{48})\frac{f^{vii}(0)}{7!} + (\frac{2}{9} - \frac{65}{32.6})\frac{f^{ix}(0)}{9!} + \ldots \quad (4.1)$$

From equation (3.2), We have

$$R_3(f) = \frac{4}{3}[f(1) - f(-1)] - \frac{1}{3}[f'(-1) + f(1)]$$

$$So,\quad \frac{4}{3}[f(1) - f(-1)] = (\frac{8}{3})[f'(0) + \frac{f'''(0)}{3!} + \frac{f^{v}(0)}{5!} + \frac{f^{vii}(0)}{7!} + \ldots]$$

$$Now\quad f(z) = f(0) + zf'(0) + \frac{z^2}{2!}f''(0) + \frac{z^3}{3!}f'''(0) + \ldots$$

$$\Rightarrow f'(z) = f'(0) + zf''(0) + \frac{z^2}{2!}f'''(0) + \frac{z^3}{3!}f^{iv}(0) + \ldots$$

$$So\quad f'(1) = f'(0) + f''(0) + \frac{f'''(0)}{2!} + \frac{f^{iv}(0)}{3!} + \frac{f^{v}(0)}{4!} + \frac{f^{vi}(0)}{5!} + \ldots$$

$$and\quad f'(-1) = f'(0) - f''(0) + \frac{f'''(0)}{2!} - \frac{f^{iv}(0)}{3!} + \frac{f^{v}(0)}{4!} - \frac{f^{vi}(0)}{5!} + \ldots$$

$Hence,\quad R_3(f)$

$$= \frac{6}{3}f'(0) + (\frac{8}{18} - \frac{1}{3})f'''(0) + (\frac{8}{360} - \frac{1}{36})f^{v}(0) + (\frac{8}{3.7!} - \frac{2}{3.6!})f^{vii}(0) + \ldots$$

$$= 2f'(0) + \frac{1}{9}f'''(0) - \frac{1}{180}f^{v}(0) + (\frac{8}{3.7!} - \frac{2}{3.6!})f^{vii}(0) + \ldots$$

Hence the error associated with the rule $R_3(f)$, which is denoted as $ER_3(f)$, and is given by

$$ER_3(f) = \int_{-1}^{1}\frac{f(z)}{z}\,dz - RS_{1/3}(f) = (\frac{2}{5.5!} + \frac{2}{3.5!})f^{v}(0) + (\frac{2}{7.7!} - \frac{8}{3.7!} + \frac{2}{3.6!})f^{vii}(0) + \ldots$$

$$ER_3(f) = (\frac{2}{225})f^{v}(0) + (\frac{2}{7.7!} - \frac{8}{3.7!} + \frac{2}{3.6!})f^{vii}(0) + \ldots \quad (4.2)$$

Now Multiplying 64 in Equation (4.1) and add to equation (4.2) we get

$$64ESR_{1/3}(f) + ER_3(f)$$

$$= 0 + f^{vii}(0)[\frac{64}{7!}(\frac{2}{7} - \frac{17}{48}) + (\frac{2}{7.7!} - \frac{8}{3.7!} + \frac{2}{3.6!})]$$

Therefore

$$64[\int_{-1}^{1}\frac{f(z)}{z}\,dz - RS_{1/3}(f)] + \int_{-1}^{1}\frac{f(z)}{z}\,dz - R_3(f) \approx 0$$

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$$\Rightarrow 65[\int_{-1}^{1} \frac{f(z)}{z}\, dz] = R_3(f) + 64RS_{1/3}(f)$$

$$\Rightarrow \int_{-1}^{1} \frac{f(z)}{z}\, dz = \frac{1}{65}R_3(f) + \frac{64}{65}RS_{1/3}(f) = MQ(f) \qquad (4.3)$$

Here $MQ(f)$ is the enhanced mixed quadrature rule.

4.1 Error Analysis

Theorem. 1

Let f(z) is analytic function in [-1,1], then the error associated with $RS_{1/3}$(f) is given by

$$ERS_{1/3}(f) = -\frac{f^{v}(0)}{60\,.\ 5!} + \left(\frac{2}{7} - \frac{17}{48}\right)\frac{f^{vii}(0)}{7!} + \left(\frac{2}{9} - \frac{65}{32.6}\right)\frac{f^{ix}(0)}{9!} + \ldots$$

From the above theorem it is established that the degree of precision of the rule $RS_{1/3}(f)$ is 4.

Theorem. 2

Let f(z) is analytic function in [-1,1], then the error associated with R_3(f) is given by

$$ER_3(f) = (\frac{2}{225})f^{v}(0) + (\frac{2}{7.7!} - \frac{8}{3.7!} + \frac{2}{3.6!})f^{vii}(0) + \ldots$$

From the above theorem it is established that the degree of precision of the rule $R_3(f)$ is 4.

Theorem. 3

Let f(z) is analytic function in [-1,1], then the error associated with $MQ(f)$ is given by

$$EMQ(f) = \frac{-44}{1365 \times 7!}f^{vii}(0) + \frac{-85}{117 \times 9!}f^{ix}(0) + \ldots$$

From the above theorem it is established that the degree of precision of the rule $MQ(f)$ is 6,which get enhanced.

4.2 Numerical Analysis

This section presents a comparative numerical analysis of the quadrature rules $RS_{1/3}(f)$, $R_3(f)$, and MQ(f). All results are shown correct up to ten decimal places and are presented through tables and graphs. Table 1 shows the numerical results obtained using the different quadrature rules, while Figure 1 illustrates the corresponding absolute errors with respect to each rule. Similarly, Table 2 and Figure 2 present the results and error comparisons for the second test integral.

$$I_1 = \int_{-1}^{1} \frac{e^z}{z}\, dz, \qquad I_2 = \int_{-1}^{1} \frac{\sin z}{z}\, dz.$$

Table 1: Comparison of quadrature rules for I_1 (10 decimal places)

Quadrature Rule	Exact Value	Approx. Value	Abs. Error
$RS_{1/3}(f)$	2.1145017508	2.1146545459	0.0001527951
$R_3(f)$	2.1145017508	2.1051494265	0.0093523243
MQ(f)	2.1145017508	2.1145083133	0.0000065625

Table 2: Comparison of quadrature rules for I_2 (10 decimal places)

Quadrature Rule	Exact Value	Approx. Value	Abs. Error
$RS_{1/3}(f)$	1.8921661407	1.8922917645	0.0001256238
$R_3(f)$	1.8921661407	1.8837210889	0.0084450518
$MQ(f)$	1.8921661407	1.8921599080	0.0000062327

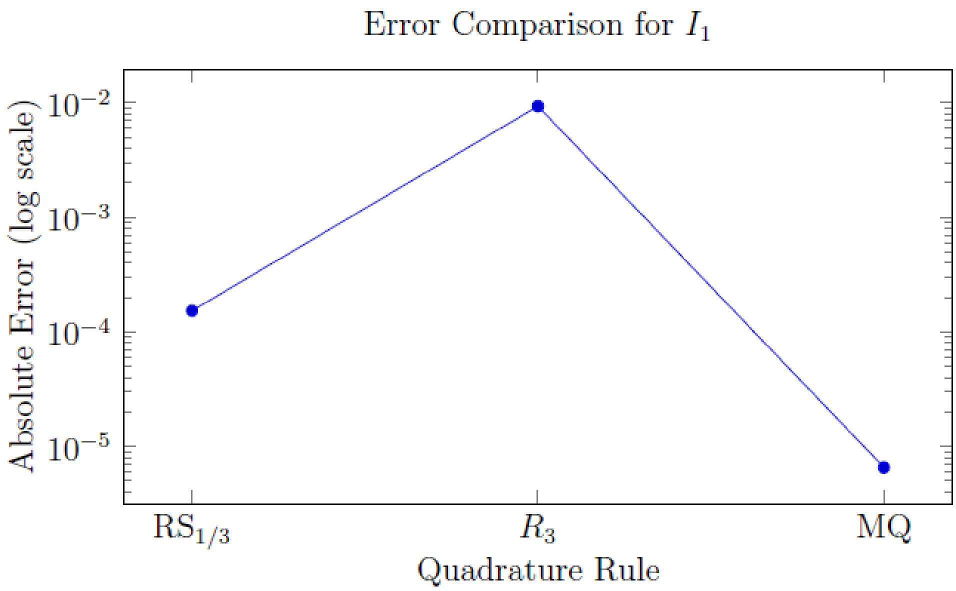

Figure 1: Absolute error of quadrature rules for I_1

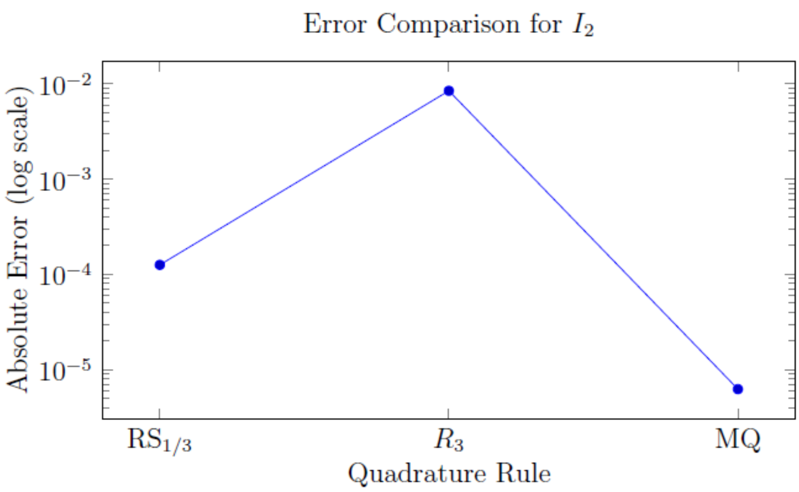

Figure 2: Absolute error of quadrature rules for I_2

4.3 Observation

From the tables and graphs, it is observed that the $\mathrm{MQ}(f)$ rule gives the smallest absolute error for both test integrals. This shows that $\mathrm{MQ}(f)$ is the most accurate among the three methods. The $\mathrm{RS}_{1/3}(f)$ rule performs fairly well but is not as precise as $\mathrm{MQ}(f)$. The $R_3(f)$ rule has the highest errors in both cases, which means it is the least effective for these types of integrals.

5. Conclusion

From the error analysis and the results shown in the tables and graphs, it is clear that the constructed higher-precision degree-6 rule $\mathrm{MQ}(f)$ provides better accuracy than its base rule. Therefore, the degree-6 $\mathrm{MQ}(f)$ rule is a more suitable and reliable choice for the numerical evaluation of Cauchy principal value integrals.

Article Information

Disclaimer (Artificial Intelligence): The author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.), and text-to-image generators have been used during writing or editing of manuscripts.

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